

TRANSIENT RESPONSE AND THE FIRST PASSAGE PROBABILITY
OF RANDOMLY EXCITED MECHANICAL SYSTEMS

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ABSTRACT

This thesis presents a detailed investigation of the first passage type of failure of vibrating systems under external random excitation. The mechanical system under consideration is idealized into a single or two-degree-of-freedom linear system and the excitation is approximated as a white noise of a certain intensity.

In the case of single-degree-of-freedom system, the Fokker-Planck equation giving the probability of the response is derived and the solution applicable to both transient and steady state determined. The probability of the system having a motion exceeding a specified critical amplitude level at any given time is determined and can be defined as the probability or reliability of the system. It is found that the first passage probability could indicate failure during the transient motion of the system especially when the damping available is small. The extension of the problem to two-degree-of-freedom systems is also considered and the solutions are derived using the Laplace transform technique.

For both cases, numerical results are presented in terms of plots using particular values for the parameters describing the system. From the results presented here, it is found that for system with low damping of the order 0.2 and less, failure can occur in the transient state itself. The effect of various system parameters and stochastic parameter on the first passage time probability for the transient and steady state of single-degree-of-freedom system is discussed. In the case of

two-degree-of-freedom system, it is noted that chance of first passage time failure in the steady state increases with time and it is sensitive to the safe limit and the variance of the system. The influence of various system parameters and stochastic parameters on the steady state first passage time probability are also discussed.

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NOMENCLATURE

A_i, B_{ij}	Coefficient of Fokker-Planck equation
a	$\frac{\beta + (\beta^2 - 4\omega_n^2)^{\frac{1}{2}}}{2}$
b	$\frac{\beta - (\beta^2 - 4\omega_n^2)^{\frac{1}{2}}}{2}$
c	Damping coefficient
D	Spectral density
$E []$	Expected value of $[]$
e	Exponential
$F(t)$	Excitation force
f_n	Natural frequency in cycles/sec.
$H(s)$	Frequency response function
$h(t)$	Impulse response function
i	Indicated imaginary value ($i^2 = -1$)
k	Spring stiffness
K_A, K_B	Safe limits for the mass m_1 and mass m_2 respectively
m	Mass of system
N_c	Average number of crossings per unit time
P_f	First passage probability
$p()$	Probability density of ()
Q_i, q_i	Coefficients of a polynomial
t	Time variable
t_s	Settling time of the system

u_i, v_i, v_i	Coefficients of a polynomial
$x(t)$	Response of the system
$\dot{x}(t)$	Time derivative of the response
y_A	Safe limit
z_1	$x_2 + ax_1$
z_2	$x_2 + bx_1$
$ (\) $	Absolute value of ()
$\alpha(t)$	Excitation force
$\sigma_{x_2}^2$	Variance of process $x(t)$
ϕ	Phase angle
ω	Frequency
ω_n	Natural frequency
ω_d	Damped frequency
ζ	Damping ratio
β	$2\zeta\omega_n$
Γ	Matrix of the variances

CHAPTER ONE

INTRODUCTION

Many engineering designs are governed by criteria that specify some critical level of amplitude which, if exceeded, will produce unacceptable results. These results might include an intolerable malfunction, or a complete failure of the system. Some common examples of such critical amplitudes include:

1. The yield or ultimate strength of the material of the system,
2. The force needed to overcome a required preload,
3. Excursion of shock mounted components which equal the allowable clearances, and
4. Maintenance schedule for industrial machinery exhibiting fluctuating vibration record.

In most practical situations, the vibration phenomenon is random in nature, and hence the design criteria must be expressed in probabilistic terms. It is then important to obtain the probabilistic information on the time when a prescribed random process $X(t)$ first passes out of a limited domain of safe operation. This is often called the first passage problem and is closely related to the calculation of the probability of "failure". With the advancement, in recent years, in the theory of stochastic process applied to engineering problems, attempts have been made to represent the vibration of a system as a random process and to use its statistical properties for failure predictions. Since in reality, the amplitudes measured fluctuate randomly with respect

to time, an analysis based on a probabilistic theory of the process involved will be more rigorous than any other conventional deterministic procedure. Very few theoretical analyses are available that would analytically determine the first passage probability of a vibratory system. Previously, there were only limited investigations that considered a probabilistic approach to failure problems in maintenance operations. One such analysis is due to Gray [1], who assumed that the vibration of a mechanical system is a stationary random process that is normally distributed with a zero mean value. The probability of failure is defined from the fact that the system ceases to function usefully when the amplitude of vibration exceeds a certain specified level. Using the theory of first passage probability of a random variable, Gray obtained certain values to represent the failure probability that is valid only for infinitely large times.

All the probabilistic quantities that describe the stochastic response of the system are, in reality, functions of time, although under stationary considerations, the mean value, variance, etc., are taken to be independent of time. The probability distribution of the response process becomes independent of time only when the time of interest approaches large values. The first passage probability calculated on the basis of such stationary response probability can only predict eventual failure of the system and cannot provide any information on the performance of the system during the transient state.

Especially for those systems with very low damping, the transient time is large and the amplitudes at transient period will affect the performance of the system more seriously than those under steady state condition. This means that it is important to obtain the stochastic characteristics of the mechanical system in the transient state as well as under steady conditions. None of the previous researchers considered the first passage probability problem during the transient motion of the system when the probabilistic properties fluctuate with time. Uhlenbeck and Wang [2] and Caughey [3] did not formulate the first passage time but only obtained the ensemble average and the variances of the response of a one-degree-of-freedom system by using the Fokker-Planck method and the impulse response technique respectively. The investigation presented here develops the first passage time probability of a single-degree-of-freedom and two-degree-of-freedom linear mechanical systems under random forces. Both the transient and the steady state conditions are taken into account for determining the first passage probability of the system. Fokker-Planck method and the Laplace transform technique are employed here to obtain the transient stochastic characteristics of these systems.

In Chapter two of this thesis, single-degree-of-freedom linear system under white noise of zero mean excitation is discussed. The first two parts of this section present the mathematical model and the corresponding Fokker-Planck equation for

single-degree-of-freedom system, using a procedure given by Ariaratnam [4]; the last two parts of this section give the solution for the response, as well as those stochastic properties of the system using the method of Uhlenbeck and Wang [2]. Chapter three discusses the first passage time probability of single-degree-of-freedom systems with a note on the settling time analysis and presents the derivation of the first passage probability for both the transient and steady states. A similar analysis for two-degree-of-freedom linear system under white noise with zero mean excitation is presented in Chapter four. In this chapter, the mathematical model is established; and the solution for the response, the first passage probability during the transient and the steady state are derived. The concluding remarks are made in Chapter five. Some of the derivations of the equations in Chapter two, three and four are presented in Appendices.

CHAPTER TWO

RESPONSE OF SINGLE-DEGREE-OF-FREEDOM SYSTEM
UNDER STOCHASTIC EXCITATION

In this section, the equations for evaluating the stochastic characteristics of the idealized system are derived.

2.1 Mathematical model of the single-degree-of-freedom linear system under random force excitation

An ideal single degree of freedom linear system, shown in Figure 2.1, consists of a mass, m , a linear spring, k , and a linear damper, c . The system is excited by a random force $F(t)$. By applying Newton's law, the equation of motion of the system is written as

$$m\ddot{X} + c\dot{X} + kX = F(t) \quad (2.1)$$

Let the undamped natural frequency of the system $\omega_n = \sqrt{k/m}$, the damping ratio $\zeta = c/2m\omega_n$, and $\alpha(t) = F(t)/m$. Then, equation (2.1) can be rewritten in the form

$$\ddot{X} + 2\zeta\omega_n\dot{X} + \omega_n^2X = \alpha(t) \quad (2.2)$$

Let the exciting random force $\alpha(t)$ be assumed to have the following properties:

1. the mean value equal to zero, i.e.

$$\langle \alpha(t) \rangle = 0 \quad (2.3)$$

2. the correlation of $\alpha(t)$ is a delta function, i.e.

$$\langle \alpha(t)\alpha(t+i) \rangle = 2D\delta(t) \quad (2.4)$$

where $2D$ is the constant spectral density or the intensity coefficient. These two properties imply that $\alpha(t)$ is a random

white noise process with a Gaussian distribution and zero mean.⁶

Since it is assumed that $\alpha(t)$ is Gaussian, and since the system is linear, it may be shown that $X(t)$ is also Gaussian [5].

For a Gaussian process, only the mean and the variances are required to characterize the response. Therefore it is only necessary to compute the stochastic, or ensemble, averages

$\langle X(t) \rangle$, $\langle \dot{X}(t) \rangle$ and the variances σ_X^2 , $\sigma_{\dot{X}}^2$ and $\sigma_{X\dot{X}}^2$ to characterize the response $X(t)$. Once these functions of time are

known, the probabilistic description of $X(t)$ will be known.

The Fokker-Planck equation technique given by Ariaratnam [4]

and Uhlenbeck and Wang }^[2] are now used to derive the response probability $p(X, \dot{X}, t)$.

2.2 Fokker-Planck equation

The equation of motion can be expressed by a set of state equations or first order differential equations. Let $X_1 = X$,

$X_2 = \dot{X}$ and $\beta = 2\zeta\omega_n$. The equation of motion (2.2) can then be expressed as

$$\begin{aligned}\dot{X}_1 &= X_2 \\ \dot{X}_2 &= -\beta X_2 - \omega_n^2 X_1 + \alpha(t)\end{aligned}\tag{2.5}$$

Since the response (X_1, X_2) represents a Markov process in the phase plane [6], the Fokker-Planck equation can be used to

find the probability density $p(X_1, X_2, t)$. In order to set

up the Fokker-Planck equation, the coefficients A_i and B_{ij} are to be evaluated first. These coefficients are defined as [2]:

$$\begin{aligned}
 A_i &= \sum a_{ik} X_k \\
 A_1 &= X_2 \\
 A_2 &= -\beta X_2 - \omega_n^2 X_1 \\
 (a_{ik}) &= \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\beta \end{bmatrix} \\
 B_{11} &= B_{12} = 0 \\
 B_{22} &= 2D
 \end{aligned} \tag{2.6}$$

Then, the Fokker-Planck equation may be expressed as follows:

$$\frac{\partial}{\partial X_1} (X_2 p) - \frac{\partial}{\partial X_2} [(\beta X_2 + \omega_n^2 X_1) p] - \frac{\partial^2}{\partial X_2^2} (D p) = \frac{\partial p}{\partial t} \tag{2.7}$$

where p is the probability density of X_1 and X_2 at time t .

The variables X_1, X_2 , are transformed in terms of Z_1, Z_2 using

$$Z_1 = X_2 + aX_1 \tag{2.8}$$

$$Z_2 = X_2 - bX_1 \tag{2.9}$$

from which

$$\begin{aligned}
 X_1 &= (Z_1 - Z_2) / (a - b) \\
 X_2 &= (aZ_2 - bZ_1) / (a - b)
 \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} a &= \frac{\beta + \sqrt{\beta^2 - 4\omega_n^2}}{2} \\ b &= \frac{\beta - \sqrt{\beta^2 - 4\omega_n^2}}{2} \end{aligned} \quad (2.11)$$

Using the following relations,

$$\begin{aligned} \frac{\partial}{\partial z_1} &= \frac{\partial}{\partial x_1} \frac{\partial x_1}{\partial z_1} + \frac{\partial}{\partial x_2} \frac{\partial x_2}{\partial z_1} \\ \frac{\partial}{\partial z_2} &= \frac{\partial}{\partial x_1} \frac{\partial x_1}{\partial z_2} + \frac{\partial}{\partial x_2} \frac{\partial x_2}{\partial z_2} \end{aligned} \quad (2.12)$$

Equation (2.7) can now be transformed into a symmetrical form with respect to the new variables z_1 and z_2 as

$$\frac{\partial p}{\partial t} = b \frac{\partial (z_1 p)}{\partial z_1} + a \frac{\partial (z_2 p)}{\partial z_2} + D \left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right)^2 p \quad (2.13)$$

2.3 Solution of the Fokker-Planck equation

In order to obtain the solution $p(X, \dot{X}, t)$ of the Fokker-Planck equation (2.7), it is necessary to solve the symmetrical form of the Fokker Planck equation (2.13). Uhlenbeck and Wang [2], used the Fourier transform technique to obtain the solution of equation (2.13) firstly in terms of z_1 and z_2 . By applying the matrix operation, the solution of equation (2.7) was obtained in terms of variables X and \dot{X} . They also calculated the ensemble mean of $\langle X(t) \rangle$ and $\langle \dot{X}(t) \rangle$, the variances σ_X^2 , $\sigma_{\dot{X}}^2$ and the covariance σ_{XX}^2 in order to define the probability

density $p(X, \dot{X}, t)$. The complete derivation of these solutions are contained in Appendix A. By defining the damped frequency ω_d as

$$\omega_d = \frac{\sqrt{4\omega_n^2 - \beta^2}}{2}$$

The ensemble means $\langle X(t) \rangle$ and $\langle \dot{X}(t) \rangle$ are obtained as follows:

$$\begin{aligned} \bar{X} = \langle X(t) \rangle &= \frac{\dot{X}_0}{\omega_d} e^{-\frac{1}{2}\beta t} \sin \omega_d t + \\ &\quad \frac{X_0}{\omega_d} e^{-\frac{1}{2}\beta t} (\omega_d \cos \omega_d t + \frac{\beta}{2} \sin \omega_d t) \\ \bar{\dot{X}} = \langle \dot{X}(t) \rangle &= \frac{\dot{X}_0}{\omega_d} e^{-\frac{1}{2}\beta t} (\omega_d \cos \omega_d t - \frac{\beta}{2} \sin \omega_d t) - \\ &\quad \frac{\omega_n^2}{\omega_d} X_0 e^{-\frac{1}{2}\beta t} \sin \omega_d t \end{aligned} \quad (2.14)$$

where X_0 and \dot{X}_0 are the initial values of $X(0)$ and $\dot{X}(0)$ respectively. The variances σ_X^2 , $\sigma_{\dot{X}}^2$ and $\sigma_{X\dot{X}}^2$ are expressed as:

$$\begin{aligned} \sigma_X^2(t) &= \langle [X(t) - \bar{X}(t)]^2 \rangle = \frac{D}{\omega_n^2 \beta} \left[1 - \frac{e^{-\beta t}}{\omega_d^2} (\omega_d^2 + \right. \\ &\quad \left. \frac{\beta^2}{2} \sin^2 \omega_d t + \frac{\beta \omega_d}{2} \sin 2\omega_d t) \right] \\ \sigma_{\dot{X}}^2(t) &= \langle [\dot{X}(t) - \bar{\dot{X}}(t)]^2 \rangle = D \left[\frac{1}{\beta} - \frac{e^{-\beta t}}{\omega_d^2} \left(\frac{\omega_d^2}{\beta} + \right. \right. \\ &\quad \left. \left. \frac{\beta}{2} \sin^2 \omega_d t - \frac{\omega_d}{2} \sin 2\omega_d t \right) \right] \end{aligned} \quad (2.15)$$

$$\begin{aligned} \sigma_{XX}^2(t) &= \langle [\dot{X}(t) - \bar{\dot{X}}(t)] [\dot{X}(t) - \bar{\dot{X}}(t)] \rangle \\ &= \frac{De^{-\beta t}}{\omega_d^2} \times \sin^2 \omega_d t \end{aligned} \quad (2.15)$$

From the stochastic properties of the system obtained above, the probability density $p(X, \dot{X}, t)$ can then be found as:

$$\begin{aligned} p(X, \dot{X}, t) &= \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)} \left(\frac{X-\bar{X}}{\sigma_X}\right)^2 + \right. \\ &\quad \left. \left(\frac{\dot{X}-\bar{\dot{X}}}{\sigma_{\dot{X}}}\right)^2 - \frac{2\rho}{\sigma_X\sigma_{\dot{X}}} (X-\bar{X})(\dot{X}-\bar{\dot{X}})\right\} \end{aligned} \quad (2.17)$$

where

$$\rho = \frac{\sigma_{XX}^2}{\sigma_X\sigma_{\dot{X}}}$$

This describes the response process of the mechanical system and is used in calculating the first passage probabilities.

2.4 Analysis of the transient stochastic properties of the system

In the previous section the stochastic properties of the system are all represented in terms of time t as shown in equations (2.14) and (2.15). From these two sets of equations, it is quite obvious that the ensemble means \bar{X} and $\bar{\dot{X}}$ and the variances σ_X and $\sigma_{\dot{X}}$ will become independent of time as time t approaches infinity. The plots of the mean values \bar{X} , $\bar{\dot{X}}$, and

and the variances σ_X and $\sigma_{\dot{X}}$ against the natural frequency are shown in Figures 2.2, 2.3, 2.4 and 2.5 respectively. From Equation (2.14), it may be seen that the magnitude of the ensemble averages depends largely on the damping ratio ζ ; the magnitudes of \bar{X} and $\bar{\dot{X}}$ increase as ζ decreases. Also, it may be noted for $\zeta > 0$, \bar{X} and $\bar{\dot{X}}$ will approach zero values as time t becomes infinite. From Equation (2.15), it is found that the magnitudes of σ_X and $\sigma_{\dot{X}}$ are influenced by the damping ratio ζ and the intensity coefficient D ; they both increase with decreasing ζ or increasing values of D . These two variances approach a constant value given by $(\sigma_X)_s = \left(\frac{D}{\omega_n^2 \beta}\right)^{\frac{1}{2}}$ and $(\sigma_{\dot{X}})_s = \left(\frac{D}{\beta}\right)^{\frac{1}{2}}$ as time t approaches infinity. The first passage probability analysis, based on the solutions presented in this chapter, is considered in the next chapter.

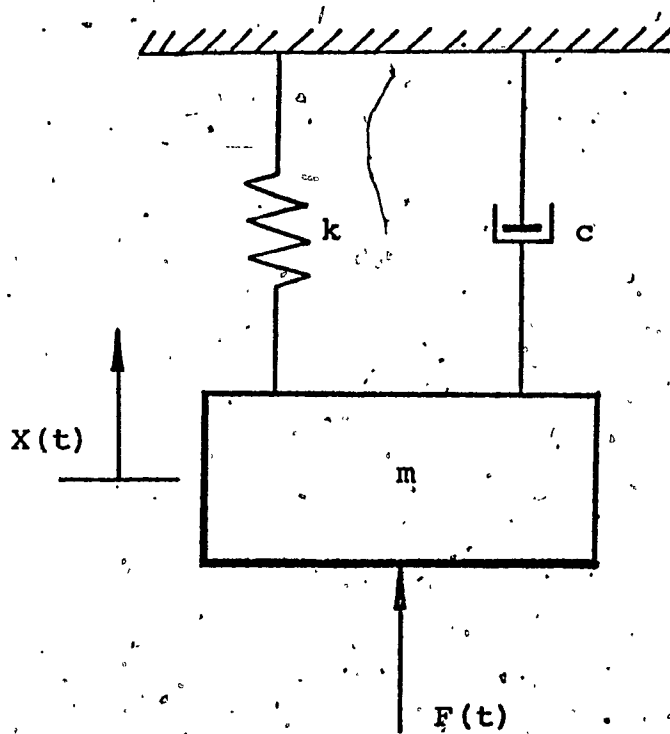


Figure 2.1 One-degree-of-freedom system.

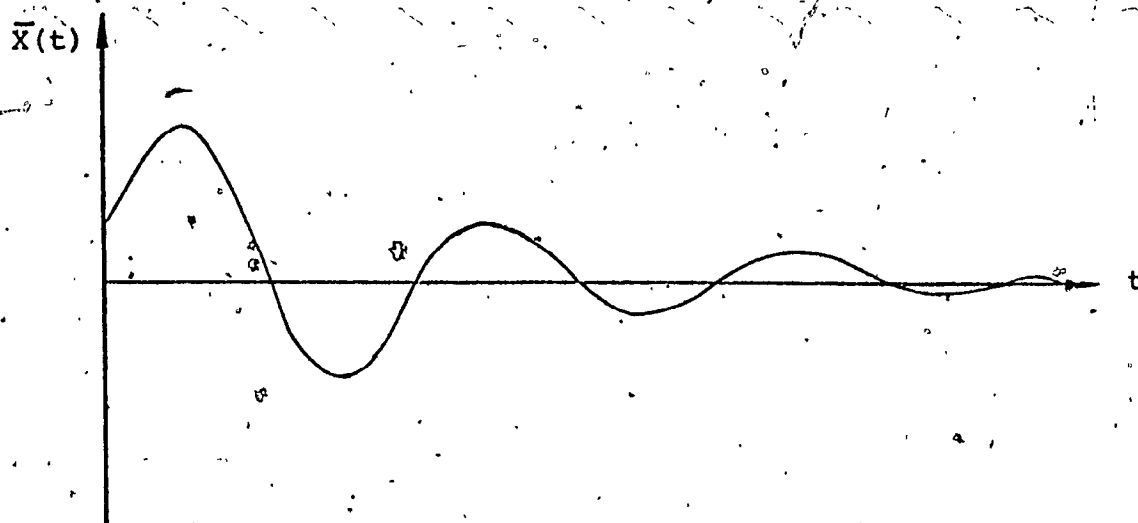


Figure 2.2 Generalized response $\bar{X}(t)$ of one degree-of-freedom linear system with $0 < \zeta < 1.0$

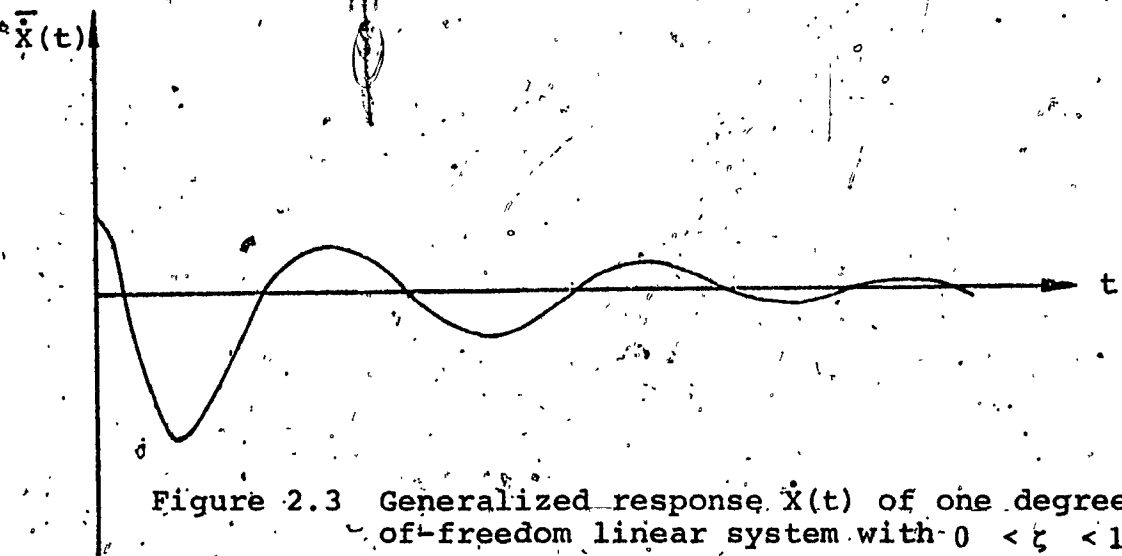


Figure 2.3 Generalized response $\dot{X}(t)$ of one degree-of-freedom linear system with $0 < \zeta < 1.0$

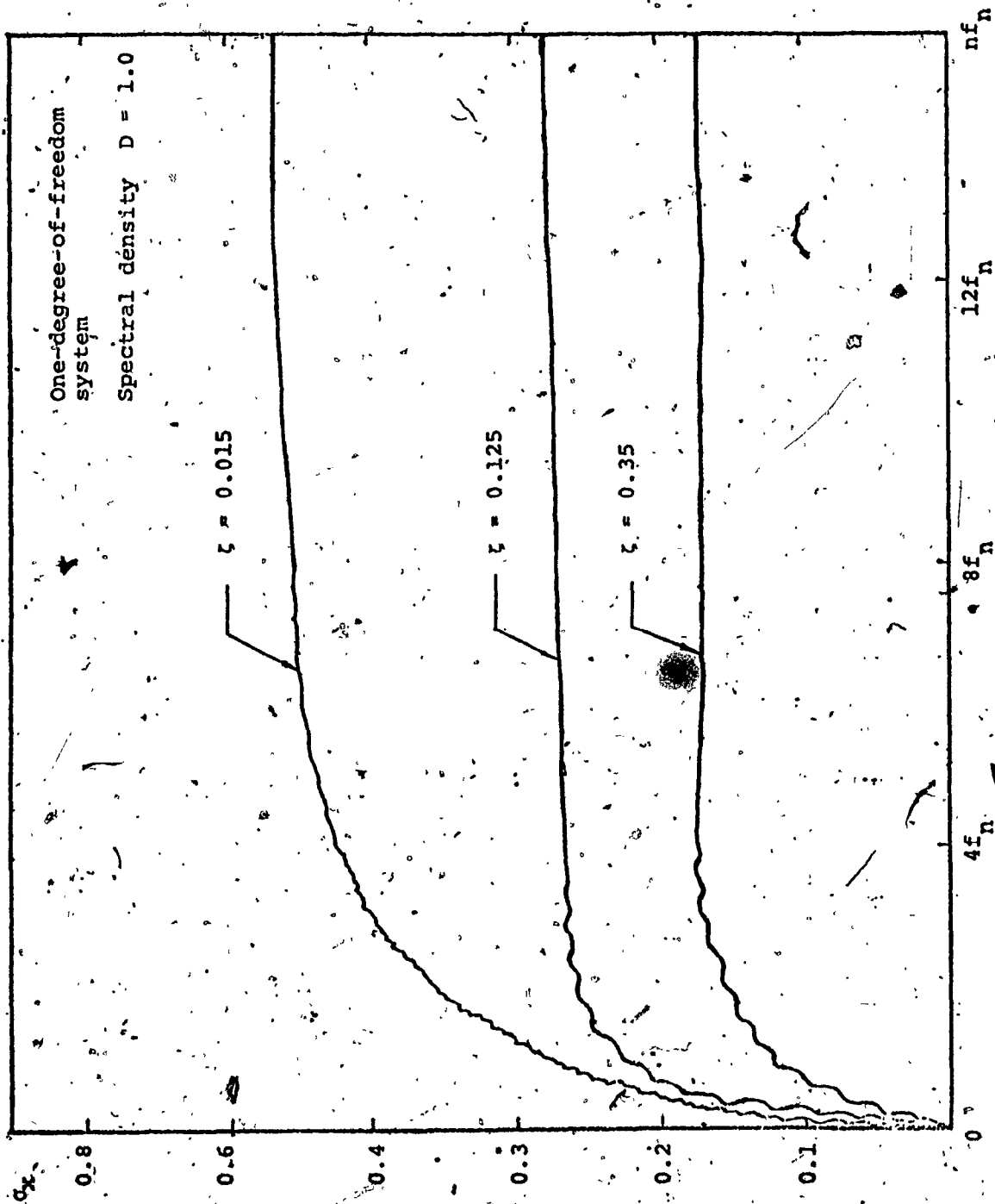


Figure 2.4 Variance of $\dot{X}(t)$ against the multiples of natural frequency of the system

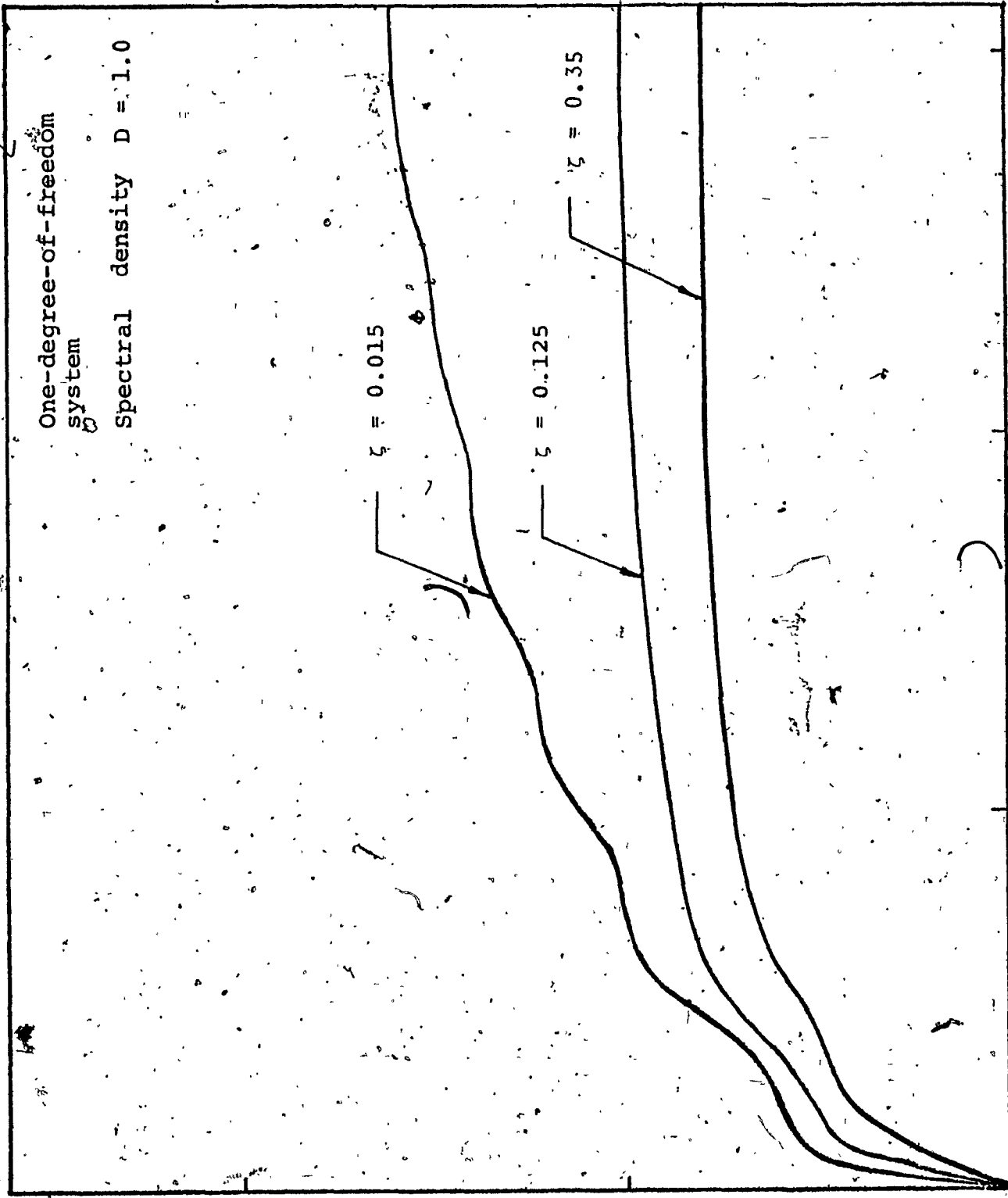


Figure 2.5 Variance of $\dot{x}(t)$ against multiples of natural frequency of the system

CHAPTER THREE

FIRST PASSAGE PROBABILITY OF SINGLE
DEGREE-OF-FREEDOM LINEAR SYSTEM

3.1 Settling time t_s of the system

In the previous chapter, the transient response of the system was considered and all the stochastic properties were represented as functions of time t . It was noticed that as the damping ratio ζ of the system increases, all these properties take a decreasing amount of time to reach their steady state values. That is, the system reaches the steady state condition in a shorter time for a larger damping ratio.

Generally speaking, the limit of the transient state (or the start of the steady state) is defined by the settling time t_s , which is the time required for the response to decrease and stay within a specified percentage of its final value.

In this investigation, the settling time (or the transient time) t_s is defined as the time required for the response to decrease to, and stay within 5% of, its initial value. Within the settling time period, the system is defined as being in a transient state, and after the settling time, the system is defined as being in a steady state condition. In some systems, the transient state is more important than the steady state because it affects the performance of the system more severely than the steady state response. On the contrary, for certain other systems, such as machine tools etc., which have large damping, the transient state is not at all significant and the steady states play the important role in determining the mechanical failure of the system. To evaluate the transition

from transient to steady state and its influence on the system, a computer program was developed to determine the settling time for single-degree-of-freedom linear system. Figure 3.1 shows the curves presenting damping ratio against settling time t_s . From this, it may be seen that for a given ω_n , as damping ratio ζ becomes smaller, the settling time t_s of the system becomes larger. That is, the system takes a longer time to settle to the steady state. Such systems are more susceptible to first passage type of failure during its transient motion and are of interest in this investigation.

3.2 Mathematical derivation of the first passage probability of single degree-of-freedom linear system

The first passage is defined as the first occurrence of a pre-selected level ky_s , experienced by a random amplitude $X(t)$, after a time interval T from an arbitrary starting time t_0 . It is the first time that $X(t)$ exceeds the amplitude ky_s , where y_s is defined as the safe-limit for operation, and k is a factor less than unity. It is now necessary to develop the equations for the probability of first passage for a transient state of a system mentioned in Chapter two.

As indicated in Figure 3.2, the curve represents the amplitude response of $X(t)$ against time t . This curve passes through $X = 0$ between the time intervals t_1 and $t_1 + \Delta t$. The intercept of this response on $X = 0$ is given by

$$t_1 - \frac{\dot{X}}{\ddot{X}}$$

where X and \dot{X} are the magnitude and slope at time $t = t_1$ respectively. The negative sign arises because $X < 0$ at time $t=t_1$. The relationship may then be expressed as

$$t_1 < t_1 - \frac{X}{\dot{X}} < t_1 + \Delta t$$

or by simplification

$$-\dot{X}\Delta t < X < 0 \quad (3.1)$$

Therefore, if the values of X and \dot{X} satisfy the inequality (3.1), then the integration of the probability density $p(X, \dot{X}, t)$ of X and \dot{X} for the ranges satisfying $0 < \dot{X} < \infty$ and $-\dot{X}\Delta t < X < 0$ will give the probability that the curve $X(t)$ passes through zero in the interval $t, t+\Delta t$. That is, the probability distribution for a crossing to occur at $X=0$ is given as

$$P(X=0; t, t+\Delta t) = \int_0^\infty \int_{-\dot{X}\Delta t}^0 p(0, \dot{X}, t) d\dot{X} dX$$

Integrating,

$$P(X=0; t, t+\Delta t) = \int_0^\infty \dot{X} p(0, \dot{X}, t) d\dot{X} \Delta t \quad (3.2)$$

This gives the probability of $X(t)$ crossing $X=0$ in the interval $t, t+\Delta t$. Similarly, the probability of crossing the level y_A in the time interval t and $t+\Delta t$ is

$$P(X=y_A; t, t+\Delta t) = \int_0^\infty \dot{X} p(y_A, \dot{X}, t) d\dot{X} \Delta t \quad (3.3)$$

In the above expression, the probability density $p(y_A, \dot{X}, t)$ has already been determined in equation (2.17). Now, let

$$K = \frac{\exp(-y_A - \bar{X})^2 / 2\sigma_X^2)}{2\pi\sigma_X\sigma_{\dot{X}}\sqrt{1-\rho^2}}$$

$$A = \bar{X} + \frac{\sigma_{\dot{X}}^2}{\sigma_X^2} (y_A - \bar{X})$$

$$\sigma_A^2 = (1-\rho^2) \sigma_{\dot{X}}^2$$

$$y_A = ky_s$$
(3.4)

Substituting equation (2.17) into (3.3), the probability of crossing $X = ky_s$ in the time interval t and $t+\Delta t$ can be expressed as

$$P(X=y_A; t, t+\Delta t) = K[\sigma_A^2 + A\sqrt{\pi/2} \sigma_A - 2\sigma_A^2 \int_0^{A/\sqrt{2}\sigma_A} u e^{-u^2} du + \sqrt{2}A\sigma_A^2 \int_0^{A/\sqrt{2}\sigma_A} e^{-u^2} du] dt$$
(3.5)

The detailed derivation of equation (3.5) is given in Appendix B.

It is now necessary to obtain the probability of the first crossing of the amplitude above a given level ky_s in a specified time interval. Let this probability be represented by P_f . The following argument may be used to derive P_f .

Let $P_0(t+\Delta t)$ represent a conditional probability that no crossing occurs about the level y_A between time t and $t+\Delta t$.

Then,

$$P_0(t+\Delta t|t) = 1 - P(X=y_A; t, t+\Delta t) \quad (3.6)$$

Again, $P_0(t+\Delta t)$ define the probability of no crossing about the level y_A up to time $t+\Delta t$, and $P_0(t)$ as the probability of no crossing up to time t . It can thus be seen that the two probabilities $P_0(t+\Delta t)$ and $P_0(t)$ are independent of each other. Therefore,

$$P_0(t+\Delta t) = P_0(t)P_0(t+\Delta t|t) \quad (3.7)$$

By substituting equation (3.6) into equation (3.7), the probability $P_0(t+\Delta t)$ is expressed as

$$P_0(t+\Delta t) = P_0(t) [1 - P(X=y_A; t, t+\Delta t)] \quad (3.8)$$

Simplifying the above equation, we have

$$P_0(t+\Delta t) - P_0(t) = -P(X=y_A; t, t+\Delta t) P_0(t)$$

or

$$\frac{\Delta P_0(t)}{P_0(t)} = -P(X=y_A; t, t+\Delta t) \quad (3.9)$$

By integrating the above equation, the probability $P_0(t)$ is expressed as

$$P_0(t) = \exp\left[-\int_0^t P(X=y_A; t, t+\Delta t) dt\right] \quad (3.10)$$

Since $P_0(t)$ is defined as the probability of no crossing up to time t , then the first passage probability P_f is

$$\begin{aligned} P_f &= 1 - P_0(t) \\ &= 1 - \exp\left\{-\int_0^t K \left[\sigma_A^2 \frac{A/\sqrt{2}\sigma_A}{\pi/2\sigma_A} - 2\sigma_A^2 \int_0^{A/\sqrt{2}\sigma_A} u \exp(-u^2) du \right. \right. \\ &\quad \left. \left. + \sqrt{2}A\sigma_A^2 \int_0^{A/\sqrt{2}\sigma_A} \exp(-u^2) du \right] dt\right\} \end{aligned} \quad (3.11)$$

In order to obtain the first passage probability P_f in Equation (3.11), the first two integrals $\int_0^{A/\sqrt{2}\sigma_A} u \exp(-u^2) du$ and $\int_0^{A/\sqrt{2}\sigma_A} \exp(-u^2) du$ must be evaluated. These two integrals are known error functions and both have finite values as u approaches infinity. A least square curve fitting technique was employed here to obtain polynomials to express these two integrals, as shown in Figures 3.3 and 3.4. These two polynomials are expressed as

$$\begin{aligned} g(u) = \int_0^u u \exp(-u^2) du &= 4.23 \times 10^{-3} - 7.978 \times 10^{-2}u \\ &+ 8.257 \times 10^{-1}u^2 - 5.394 \times 10^{-1}u^3 + 8.856 \times 10^{-2}u^4 \\ &+ 1.663 \times 10^{-2}u^5 - 4.611 \times 10^{-3}u^6 \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} h(u) = \int_0^u \exp(-u^2) du &= 0.774 \times 10^{-3} + 0.98u + 0.128u^2 \\ &- 0.659u^3 + 0.379u^4 - 0.903 \times 10^{-1}u^5 + 0.806 \times 10^{-2}u^6 \end{aligned} \quad (3.13)$$

The reason for expressing these two integrals as sixth order polynomials is that the residue errors of the sixth order polynomials for these two integrals are found to be 0.001 for $g(u)$ and 0.002 for $h(u)$, and such residue errors are sufficiently

small that they can be neglected for computing purposes. The range for these two polynomials is between 0 and 3.0; if the upper limit of the integrals is bigger than 3.0, the integrals will be expressed as $g(u > 3) = 0.50$ and $h(u > 3) = 0.8862$.

Using these polynomial representations, the first passage probability P_f in equation (3.11) is expressed as

$$P_f = 1 - \exp \left\{ - \int_0^t K \left[\sigma_A^2 + A\sqrt{\pi/2}\sigma_A - 2\sigma_A g(A/\sqrt{2}\sigma_A) + \sqrt{2}A\sigma_A^2 h(A/\sqrt{2}\sigma_A) \right] dt \right\} \quad (3.14)$$

The first passage probability P_f about level $X = y_A$, in equation (3.14), is now expressed in terms of time t and can be evaluated if time t is specified. A numerical technique is applied to evaluate the integral in equation (3.14) and the first passage probability is expressed as

$$P_f = 1 - \exp \left\{ - \sum_{n=0}^{n\Delta t} K \left[\sigma_A^2 + A\sqrt{\pi/2}\sigma_A - 2\sigma_A g(A/\sqrt{2}\sigma_A) + \sqrt{2}A\sigma_A^2 h(A/\sqrt{2}\sigma_A) \right] \Delta t \right\} \quad (3.15)$$

This expression is valid for both transient and steady state motion of the system.

3.3 The first passage probability during the transient state and the steady state

In a linear vibratory system, if the damping ratio is small, the amplitude of response of the system will be large, and the first passage probability about a certain

level y_A will be large even for a short operating time t . But as the damping ratio increases, the amplitude of response decreases rapidly and the first passage probability about a certain level y_A will be small even for a long period of time t . Figure 3.5 gives the curves of first passage probability against the multiple of natural frequency for different values of the damping ratio and the safe level y_A . The first passage probability for an elapse of certain time t increases as the damping ratio ζ decreases, or the safe level y_A decreases, and vice versa.

In the previous chapter, the settling time t_s was defined to distinguish between transient state and steady state conditions. By using this definition of settling time, the first passage probability during transient state and steady state can be obtained. Figure 3.6 shows the curves for the first passage probability against the damping ratio at settling time. This indicates the first passage probability at the end of transient motion and shows the effect of transient state of the system on its performance. For smaller damping ratios, the transient state dominates the first passage time probability but for a larger damping ratio, the transient first passage probability can be neglected and the failure of the system will be determined solely by the steady state motion.

In the steady state condition, the first passage probability can be simplified by letting time t equal infinity.

From equation (3.14), if $t \rightarrow \infty$, the whole expression reduces to

$$(P_f)_s = 1 - \exp \left[-\frac{\sigma_{\dot{x}_s}}{2\pi\sigma_{x_s}} T e^{-y_A^2/2\sigma_{x_s}^2} \right] \quad (3.16)$$

The variances σ_{x_s} and $\sigma_{\dot{x}_s}$ are the variances at steady state condition and they are expressed as

$$\sigma_{x_s} = \sqrt{D/\omega_n^2\beta}$$

and

(3.17)

$$\sigma_{\dot{x}_s} = \sqrt{D/\beta}$$

By defining the average number of crossings per unit time as,

$$\begin{aligned} N_c &= \frac{\sigma_{\dot{x}_s}}{2\pi\sigma_{x_s}} \times e^{-y_A^2/2\sigma_{x_s}^2} \\ &= \frac{\omega_n}{2\pi} e^{-K^2/2} \\ &= f_n \exp(-K^2/2) \end{aligned} \quad (3.18)$$

where f_n is the natural frequency of the system in cycles/sec., and K is the ratio between the safe level y_A and variance σ_{x_s} , the first passage probability at steady state can be rewritten as

$$(P_f)_s = 1 - \exp(-f_n T e^{-K^2/2}) \quad (3.19)$$

Figure 3.7 gives the curves of the average number of crossings against the values K for different values of natural frequencies f_n . N_c increases as f_n increases or K decreases. The first passage probability $(P_f)_s$ at steady state versus the number of cycles for different values of K are plotted in Figure 3.8. The first passage probability $(P_f)_s$ increases as the value K decreases.

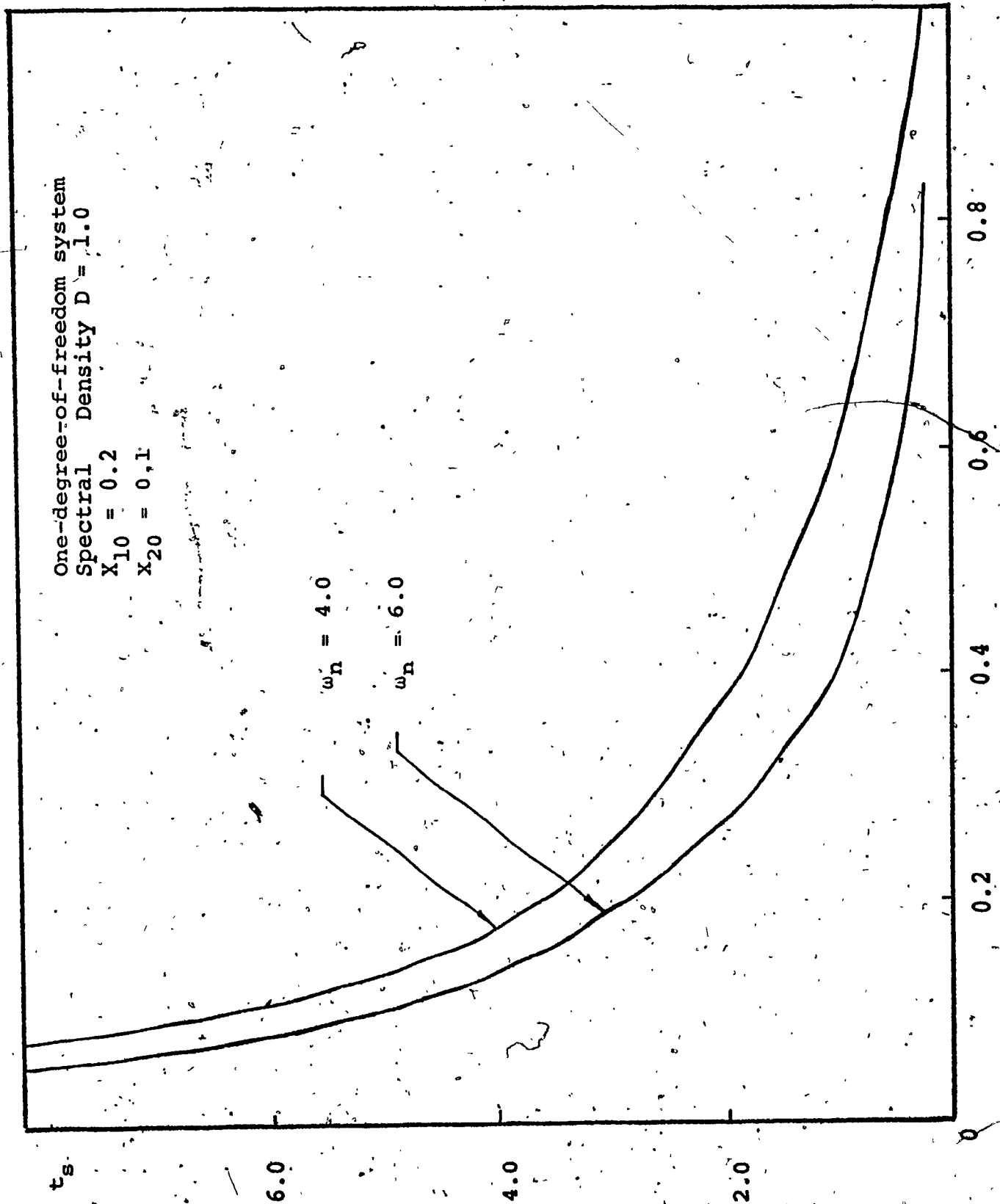


Figure 3.1 Settling time against damping ratio of one-degree-of-freedom linear system

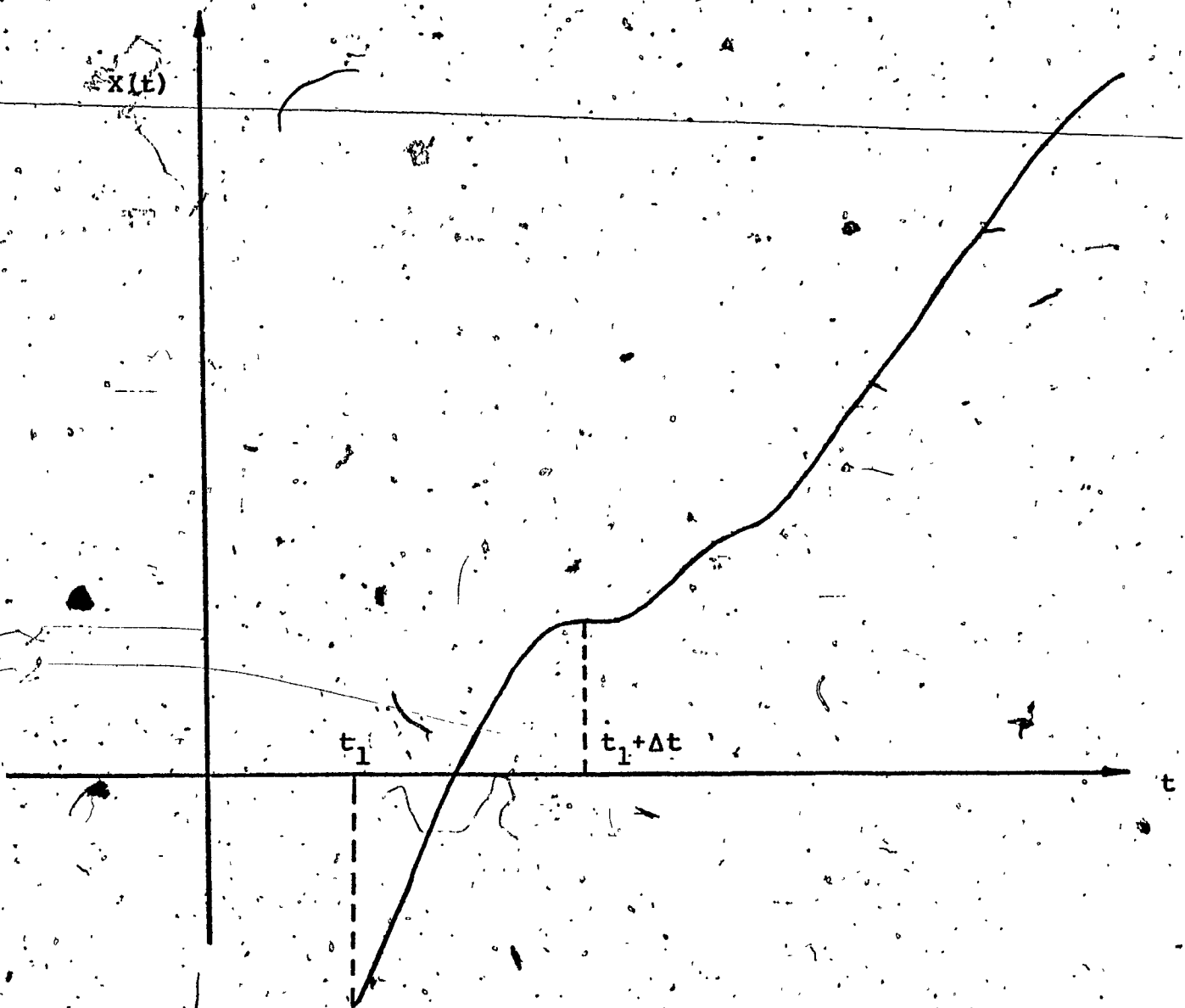


Figure 3.2 The response $X(t)$ crosses the level $X = 0$ between time t_1 and time $t_1 + \Delta t$

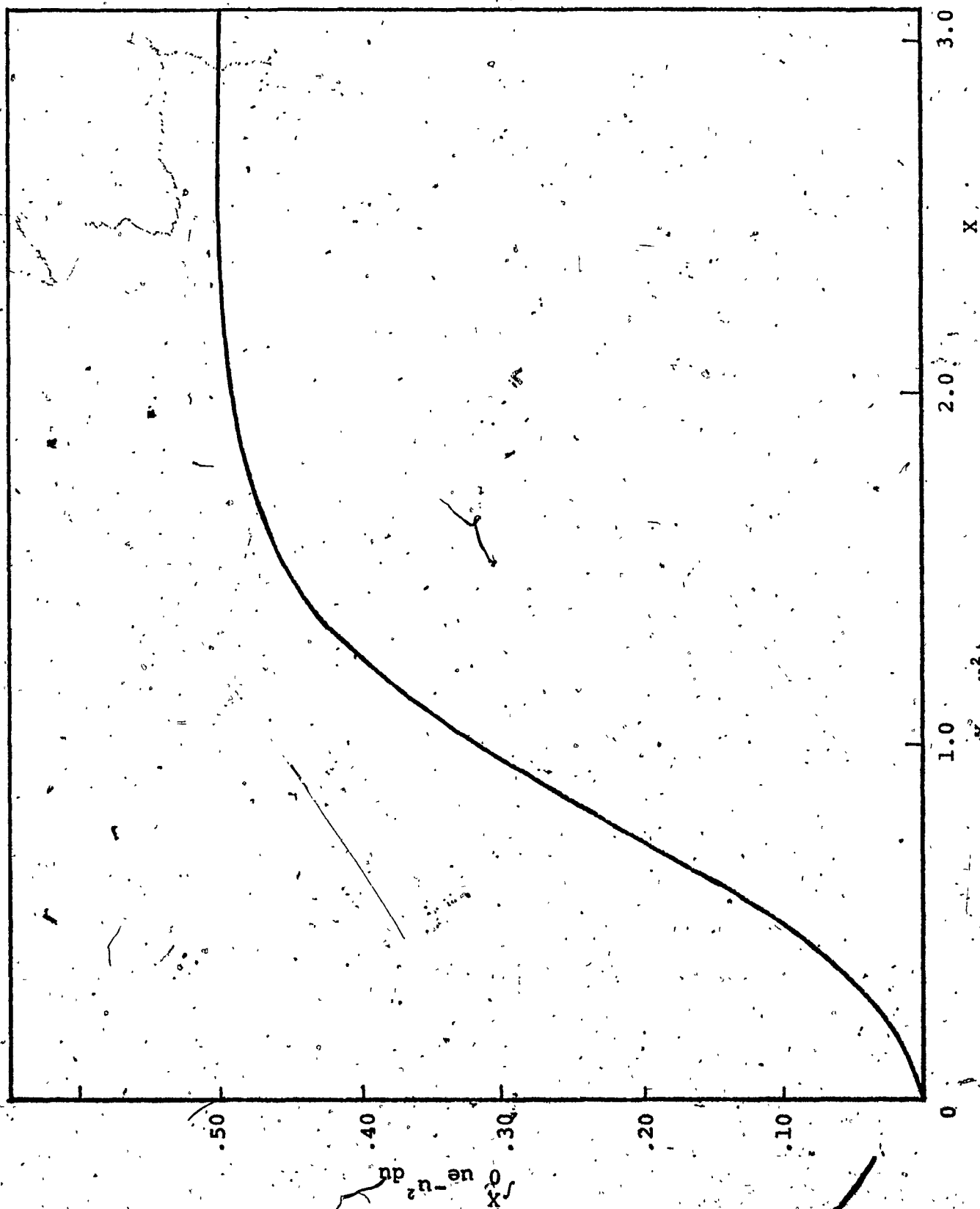


Figure 3.3 The integral $\int_0^x x^2 e^{-u^2} du$ against x

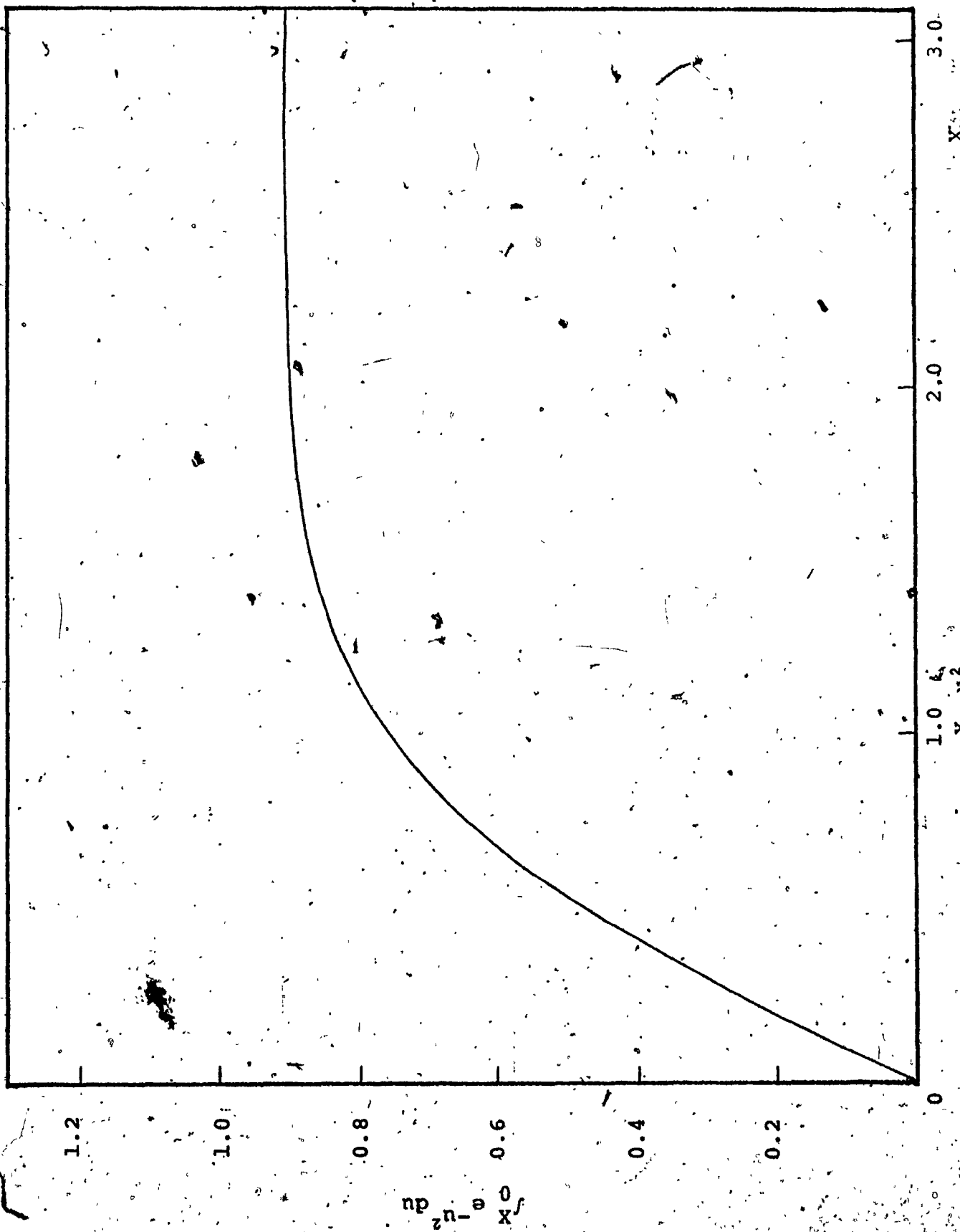


Figure 3.4 The integral $\int_0^X e^{-u^2} du$ against X

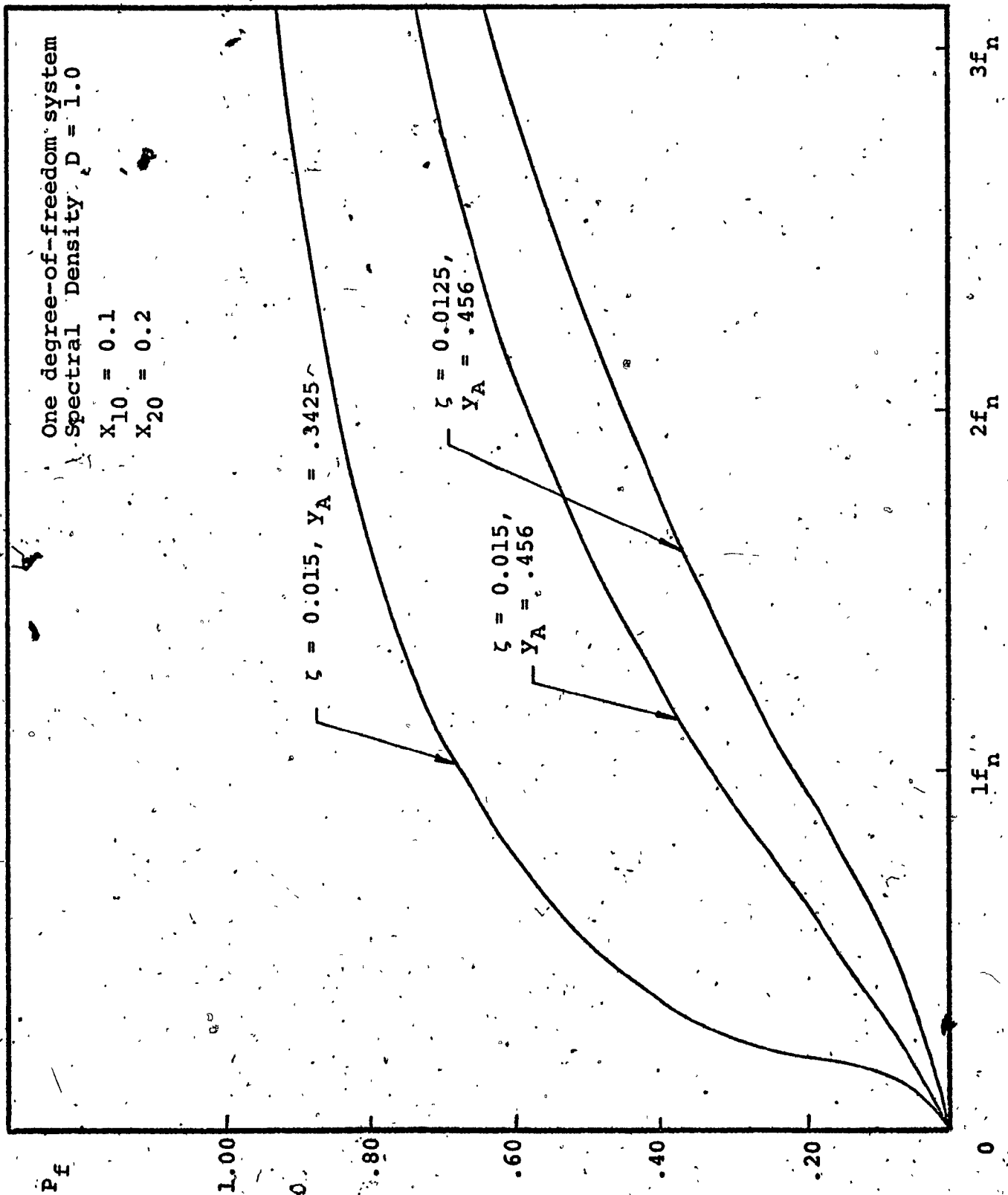


Figure 3.5 Transient first passage probability against multiples of natural frequency

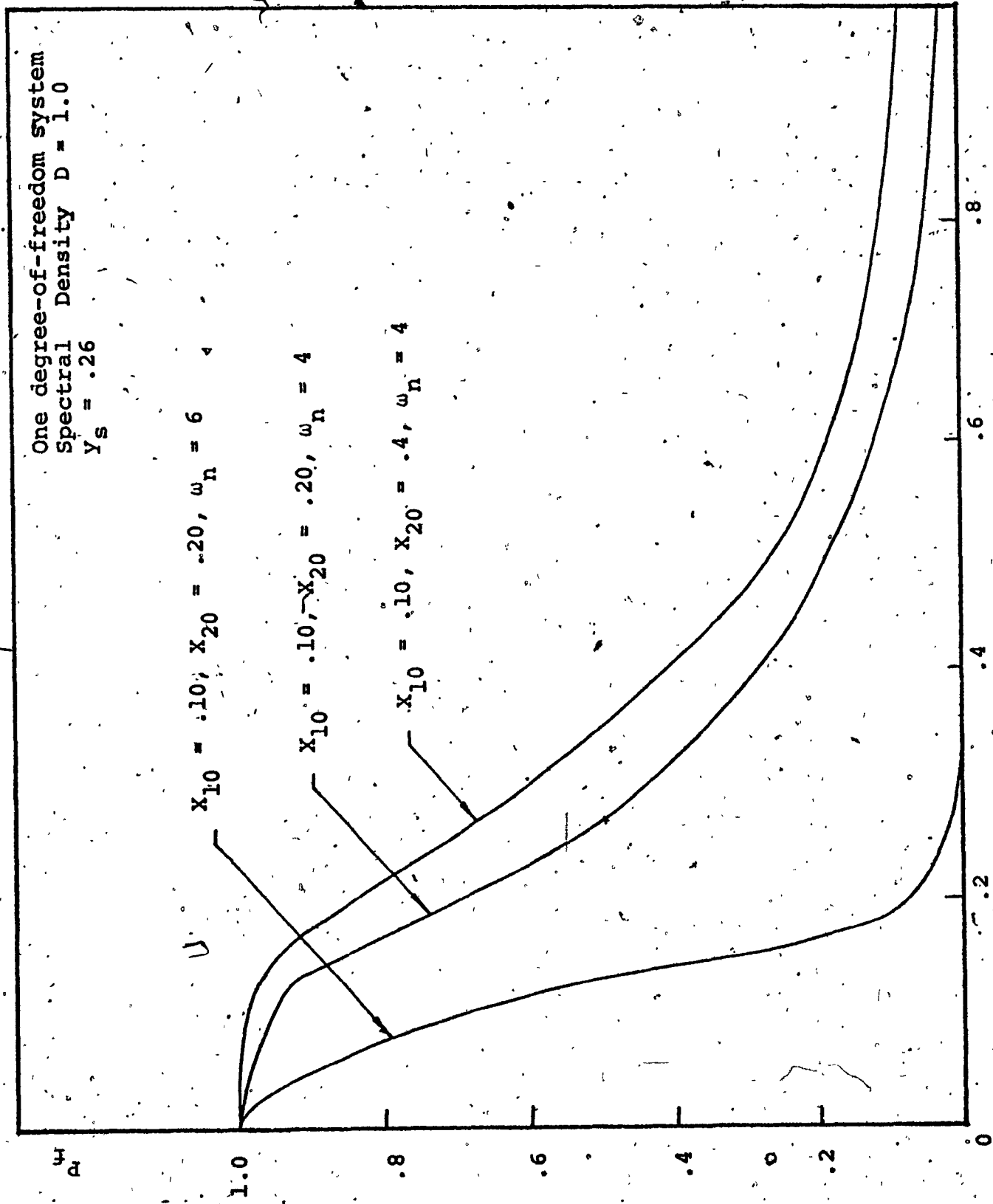


Figure 3.6 The first passage probability against damping ratio at settling time

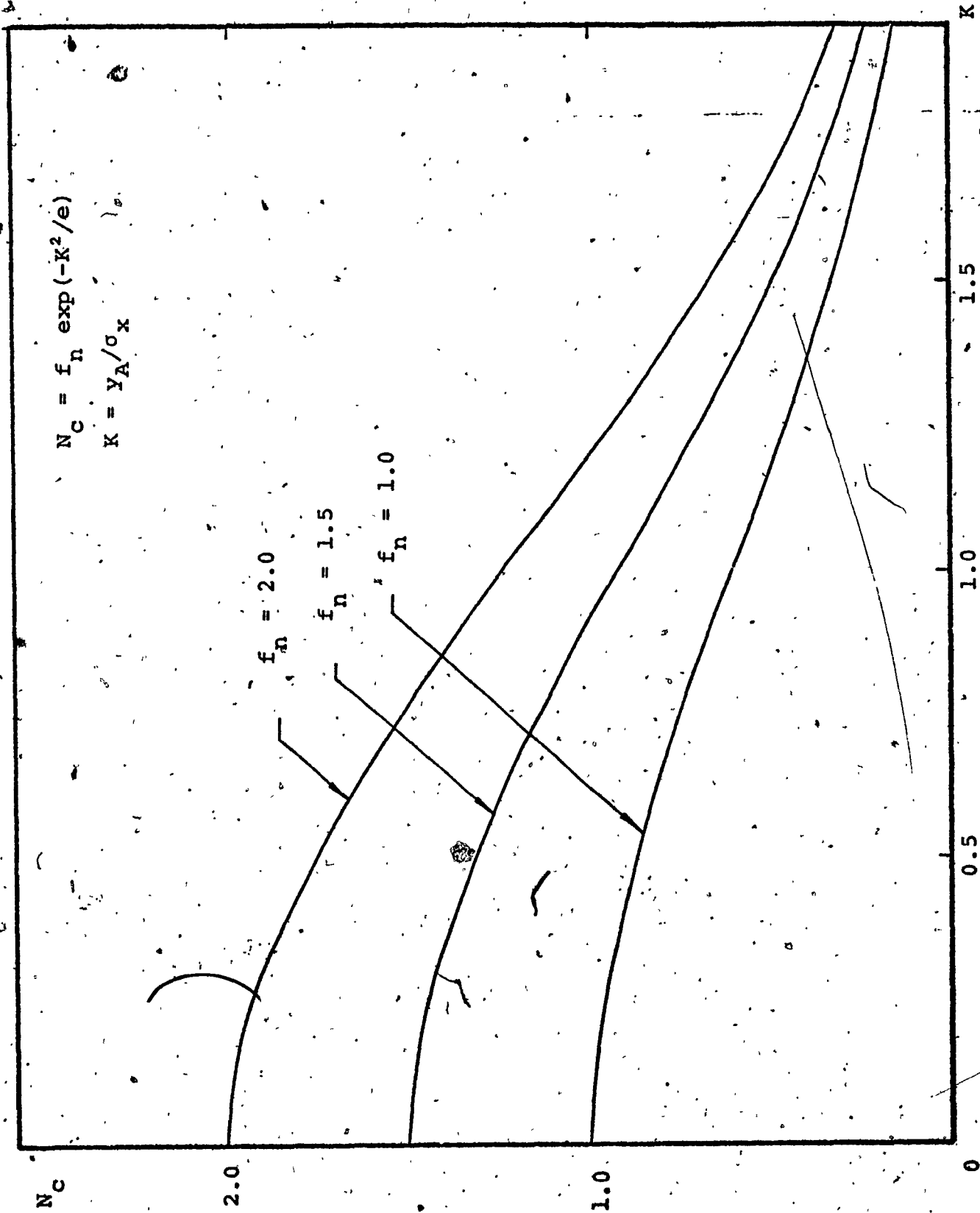


Figure 3.7 Average number of crossings against the constant K at steady state condition

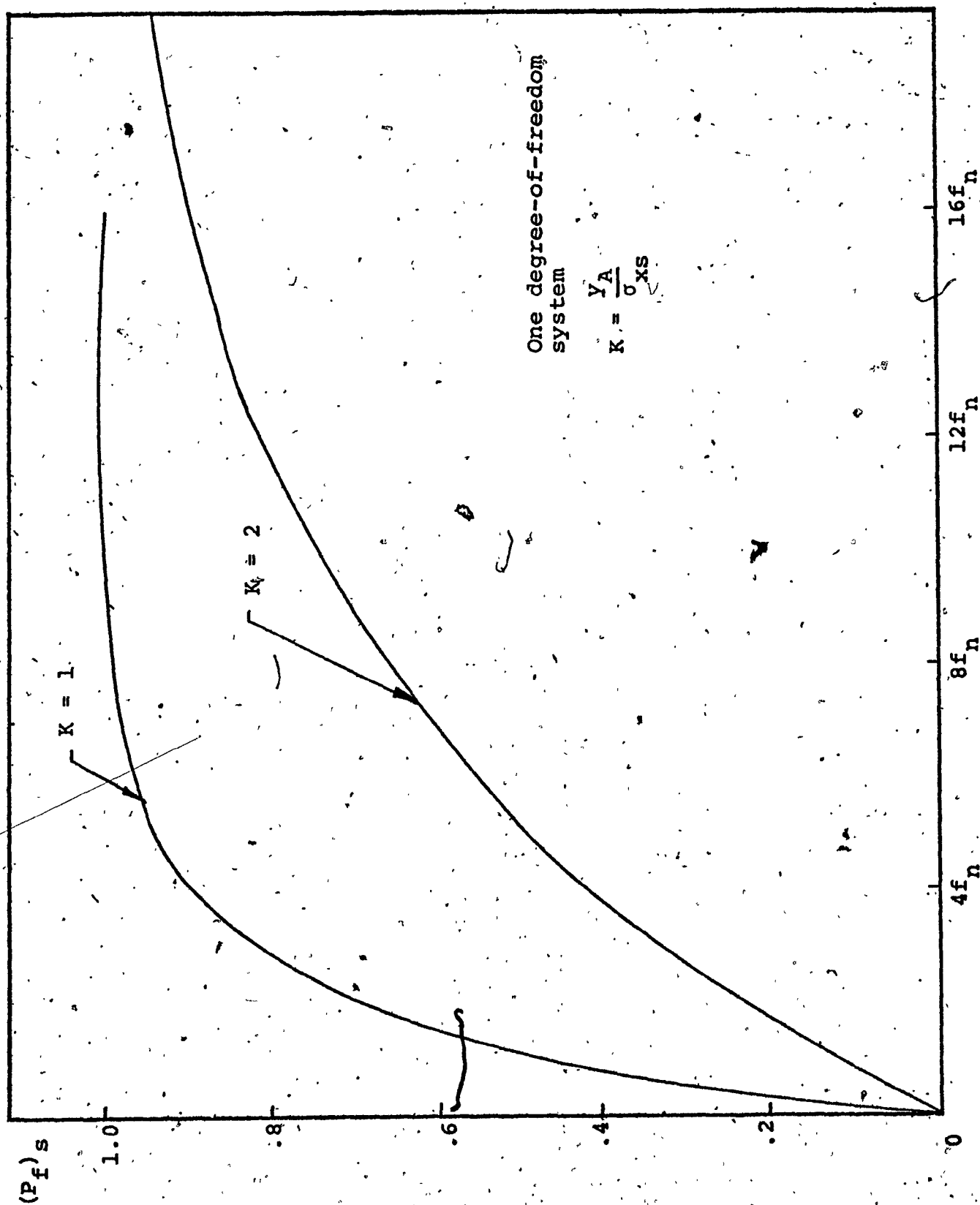


Figure 3.8 Stationary first passage probability against multiples of natural frequency of the system

CHAPTER FOUR

STOCHASTIC RESPONSE OF TWO-DEGREE-OF-FREEDOM

LINEAR SYSTEMS

4.1 Introduction

In the two previous chapters, the first passage probability of single-degree-of-freedom system under stationary random excitation was investigated. In reality, not all the systems can be modeled into a single-degree-of-freedom system. In this chapter, the investigation of the first passage probability is extended for a mechanical system with two-degree-of-freedom subjected to random excitation.

4.2 Frequency Response of a Two-Degree-Of-Freedom linear system

An idealized two-degree-of-freedom system as shown in fig. 4.1 is considered. Using the d'Alembert principle, the equation of motion may be written as

$$\begin{aligned} m_1 \ddot{X}_1 + (c_1 + c_2) \dot{X}_1 - c_2 \dot{X}_2 + (k_1 + k_2) X_1 - k_2 X_2 &= F(t) \\ m_2 \ddot{X}_2 + c_2 \dot{X}_2 - c_2 \dot{X}_1 + k_2 X_2 - k_2 X_1 &= 0 \end{aligned}$$

where

m_1, m_2 : masses of oscillators

c_1, c_2 : constants of viscous dampers

k_1, k_2 : constants of springs

$F(t)$: random excitation on m_1

X_1, X_2 : displacements of m_1 and m_2

In chapter 2, the analysis for the response of one-degree-of-freedom linear mechanical system was calculated using the Fokker Planck equation technique. A similar approach for two-degree-of-freedom system becomes tedious due to the difficulty of obtaining the coefficients A_i, B_{ij} and of solving the fourth

order partial differential equation with stochastic coefficients. Hence the problem is considered here in the frequency domain. Using the Laplace transform with initial conditions

$$\begin{aligned}x_1(0) &= x_{10} \\ \dot{x}_1(0) &= \dot{x}_{10} \\ x_2(0) &= x_{20} \\ \dot{x}_2(0) &= \dot{x}_{20}\end{aligned}\tag{4.2}$$

The two equations of motion can be written as

$$\begin{aligned}[m_1 s^2 + (c_1 + c_2)s + (k_1 + k_2)]X_1(s) - (c_2 s + k_2)X_2(s) \\ = m_1 s x_{10} + m_1 \dot{x}_{10} + (c_1 + c_2)x_{10} - c_2 x_{20} + F(s)\end{aligned}\tag{4.3a}$$

$$\begin{aligned}(m_2 s^2 + c_2 s + k_2)X_2(s) - (c_2 s + k_2)X_1(s) \\ = m_2 s x_{20} + m_2 \dot{x}_{20} + c_2 x_{20} - c_2 \dot{x}_{10}\end{aligned}\tag{4.3b}$$

Rearranging and solving equations (4.3a) and (4.3b),

$$\begin{aligned}x_1(s) = \frac{(s^3 + a_1 s^2 + b_1 s + b_2)x_{10} + (s^2 + d_1 s + d_2)\dot{x}_{10} + e_1 x_{20} + (g_1 s + e_1)\dot{x}_{20}}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \\ + \frac{(s^2 + d_1 s + d_2)F(s)/m_1}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}\end{aligned}\tag{4.4a}$$

$$\begin{aligned}x_2(s) = \frac{(d_2 s + p_1)x_{10} + (d_1 s + d_2)\dot{x}_{10} + (s^3 + a_1 s^2 + n_1 s + n_2)x_{20} + (s^2 + l_1 s + l_2)\dot{x}_{20}}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \\ + \frac{(d_1 s + d_2)F(s)/m_1}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}\end{aligned}\tag{4.4b}$$

where

$$a_1 = \frac{c_1}{m_1} + \frac{c_2}{m_2} + \frac{c_2}{m_1}$$

$$a_2 = \frac{k_1}{m_1} + \frac{k_2}{m_2} + \frac{k_2}{m_1} + \frac{c_1 c_2}{m_1 m_2}$$

$$a_3 = \frac{c_1 k_2}{m_1 m_2} + \frac{c_2 k_1}{m_1 m_2}$$

$$a_4 = \frac{k_1 k_2}{m_1 m_2}$$

$$b_1 = \frac{k_2}{m_2} + \frac{c_1 c_2}{m_1 m_2}$$

$$b_2 = \frac{c_1 k_2}{m_1 m_2}$$

$$d_1 = \frac{c_2}{m_2}$$

$$d_2 = \frac{k_2}{m_2}$$

$$e_1 = \frac{k_2}{m_1}$$

$$g_1 = \frac{c_2}{m_1}$$

$$p_1 = \frac{k_2 c_1}{m_1 m_2} + \frac{k_1 c_2}{m_1 m_2}$$

$$n_1 = \frac{k_1}{m_1} + \frac{k_2}{m_2} + \frac{c_1 c_2}{m_1 m_2}$$

$$n_2 = \frac{k_1 c_2}{m_1 m_2}$$

$$l_1 = \frac{c_1}{m_1} + \frac{c_2}{m_2}$$

$$l_2 = \frac{k_1}{m_1} + \frac{k_2}{m_2}$$

(4.5)

Let the forcing function be

$$\alpha(t) = F(t)/m_1$$

or $\alpha(s) = F(s)/m_1$

(4.6)

The transfer functions $H_1(s)$ and $H_2(s)$ giving the response $X_1(s)$ and $X_2(s)$ may be expressed as

$$H_1(s) = \frac{s^2 + d_1 s + d_2}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \quad (4.7a)$$

$$H_2(s) = \frac{d_1 s + d_2}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \quad (4.7b)$$

Then the impulse response of $X_1(t)$ and $X_2(t)$ can be obtained by taking the inverse Laplace transform of eqs (4.7a) and (4.7b) as

$$h_1(t) = L^{-1} [H_1(s)] \quad (4.8a)$$

$$h_2(t) = L^{-1} [H_2(s)] \quad (4.8b)$$

The complete response of $X_1(t)$ and $X_2(t)$ can be obtained by taking the inverse Laplace transform of equations (4.4a) and (4.4b). That is,

$$X_1(t) = L^{-1} \left[\frac{(s^3 + a_1 s^2 + b_1 s + b_2) X_{10} + (s^2 + d_1 s + d_2) X_{10} + e_1 s X_{20} + (g_1 s + e_1) X_{20}}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \right] + \int_0^t h_1(t-\tau) \alpha(\tau) d\tau \quad (4.9a)$$

$$x_2(t) = L^{-1} \left[\frac{(d_2 s + p_1) x_{10} + (d_1 s + d_2) \dot{x}_{10} + (s^3 + a_1 s^2 + n_1 s + n_2) x_{20} + (s^2 + l_1 s + l_2) \dot{x}_{20}}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \right]$$

$$+ \int_0^t h_2(t-\tau) \alpha(\tau) d\tau$$

(4.9b)

By defining

$$\gamma_1 = x_{10}$$

$$\gamma_2 = a_1 x_{10} + \dot{x}_{10}$$

$$\gamma_3 = b_1 x_{10} + d_1 \dot{x}_{10} + e_1 x_{20} + g_1 \dot{x}_{20}$$

$$\gamma_4 = b_2 x_{10} + d_2 \dot{x}_{10} + e_2 x_{20} + g_2 \dot{x}_{20}$$

and

$$q_1 = x_{20}$$

$$q_2 = a_1 x_{20} + \dot{x}_{20}$$

$$q_3 = n_1 x_{20} + d_2 x_{10} + d_1 \dot{x}_{10} + l_1 \dot{x}_{20}$$

$$q_4 = p_2 x_{20} + p_1 x_{10} + d_2 \dot{x}_{10} + l_2 \dot{x}_{20}$$

Then, equations (4.9a) and (4.9b) may be simplified as

$$x_1 = L^{-1} \left[\frac{\gamma_1 s^3 + \gamma_2 s^2 + \gamma_3 s + \gamma_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \right] + \int_0^t h_1(t-\tau) \alpha(\tau) d\tau \quad (4.10a)$$

$$x_2 = L^{-1} \left[\frac{q_1 s^3 + q_2 s^2 + q_3 s + q_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \right] + \int_0^t h_2(t-\tau) \alpha(\tau) d\tau \quad (4.10b)$$

4.3 Mean value of response

Suppose the forcing function $\alpha(t)$ has the same properties as that described in chapter 2. That is,

$$\langle \alpha(t) \rangle = 0$$

$$\langle \alpha(t-\tau) \alpha(\tau) \rangle = 2D\delta(t) \quad (4.11)$$

By defining the mean value of $X_1(t)$ as $\eta_1(t)$, from equation (4.10a),

$$\begin{aligned}\eta_1(t) &= \langle X_1(t) \rangle \\ &= \langle L^{-1} \left[\frac{\gamma_1 s^3 + \gamma_2 s^2 + \gamma_3 s + \gamma_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \right] + \int_0^t h_1(t-\tau) \alpha(\tau) d\tau \rangle \\ &= L^{-1} \left[\frac{\gamma_1 s^3 + \gamma_2 s^2 + \gamma_3 s + \gamma_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \right] + \langle \int_0^t h_1(t-\tau) \alpha(\tau) d\tau \rangle \\ &= L^{-1} \left[\frac{\gamma_1 s^3 + \gamma_2 s^2 + \gamma_3 s + \gamma_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \right] + \int_0^t h_1(t-\tau) \langle \alpha(\tau) \rangle d\tau\end{aligned}\quad (4.12)$$

Since the mean value of $\alpha(t)$ is zero, the mean value $\eta_1(t)$ may be written as

$$\eta_1(t) = L^{-1} \left[\frac{\gamma_1 s^3 + \gamma_2 s^2 + \gamma_3 s + \gamma_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \right] \quad (4.13a)$$

Similary, the mean value of $X_2(t)$ may be expressed as

$$\eta_2(t) = L^{-1} \left[\frac{q_1 s^3 + q_2 s^2 + q_3 s + q_4}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \right] \quad (4.13b)$$

The denominator of equation (4.13a) can be factorized into two second order polynomials, that is,

$$\begin{aligned}s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 \\ = [(s + \zeta_1 \beta_1)^2 + \omega_1^2] [(s + \zeta_2 \beta_2)^2 + \omega_2^2]\end{aligned}\quad (4.14)$$

The above factorization is done by means of a numerical technique which is described in appendix C.

Then equation (4.13) can be rewritten as

$$\begin{aligned} \eta_1(t) &= L^1 \left[r_1 \frac{s + \zeta_1 \beta_1}{(s + \zeta_1 \beta_1)^2 + \omega_1^2} + r_2 \frac{\omega_1^2}{(s + \zeta_1 \beta_1)^2 + \omega_1^2} + \right. \\ &\quad \left. r_3 \frac{s + \zeta_2 \beta_2}{(s + \zeta_2 \beta_2)^2 + \omega_2^2} + r_4 \frac{\omega_2^2}{(s + \zeta_2 \beta_2)^2 + \omega_2^2} \right] \\ &= \exp(-\zeta_1 \beta_1 t) [r_1 \cos \omega_1 t + r_2 \sin \omega_1 t] + \\ &\quad \exp(-\zeta_2 \beta_2 t) [r_3 \cos \omega_2 t + r_4 \sin \omega_2 t] \\ &= Q_1 \exp(-\zeta_1 \beta_1 t) \sin(\omega_1 t + \phi_1) + Q_2 \exp(-\zeta_2 \beta_2 t) \sin(\omega_2 t + \phi_2) \end{aligned} \quad (4.15a)$$

where

$$Q_1 = \sqrt{r_1^2 + r_2^2}$$

$$Q_2 = \sqrt{r_3^2 + r_4^2}$$

$$\phi_1 = \tan^{-1}(r_2/r_1)$$

$$\phi_2 = \tan^{-1}(r_4/r_3)$$

In the same manner, equation (4.13b) is rewritten

$$\begin{aligned} \eta_2(t) &= \exp(-\zeta_1 \beta_1 t) [u_1 \cos \omega_1 t + u_2 \sin \omega_1 t] + \\ &\quad \exp(-\zeta_2 \beta_2 t) [u_3 \cos \omega_2 t + u_4 \sin \omega_2 t] \\ &= V_1 \exp(-\zeta_1 \beta_1 t) \sin(\omega_1 t + \psi_1) + V_2 \exp(-\zeta_2 \beta_2 t) \sin(\omega_2 t + \psi_2) \end{aligned}$$

where

$$V_1 = \sqrt{u_1^2 + u_2^2}$$

$$V_2 = \sqrt{u_3^2 + u_4^2}$$

$$\psi_1 = \tan^{-1}(u_2/u_1)$$

$$\psi_2 = \tan^{-1}(u_4/u_3)$$

(4.15b)

The mean value of the velocities $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are equal to the derivative of $\eta_1(t)$ and $\eta_2(t)$. That is

$$\dot{\eta}_1(t) = \frac{d\eta_1(t)}{dt} = Q_3 \exp(-\zeta_1 \beta_1 t) \cos(\omega_1 t + \phi_3) + Q_4 \exp(-\zeta_2 \beta_2 t) \cos(\omega_2 t + \phi_4)$$

where

$$\begin{aligned} Q_3 &= \sqrt{(r_2 \omega_1 - r_1 \zeta_1 \beta_1)^2 + (r_1 \omega_1 + r_2 \zeta_1 \beta_1)^2} \\ Q_4 &= \sqrt{(r_4 \omega_2 - r_3 \zeta_2 \beta_2)^2 + (r_3 \omega_2 + r_4 \zeta_2 \beta_2)^2} \\ \phi_3 &= \tan^{-1} \frac{r_1 \omega_1 + r_2 \zeta_1 \beta_1}{r_1 \zeta_1 \beta_1 - r_2 \omega_1} \\ \phi_4 &= \tan^{-1} \frac{r_3 \omega_2 + r_4 \zeta_2 \beta_2}{r_3 \zeta_2 \beta_2 - r_4 \omega_2} \end{aligned} \quad (4.16a)$$

$$\dot{\eta}_2(t) = d\eta_2(t)/dt$$

$$= V_3 \exp(-\zeta_1 \beta_1 t) \cos(\omega_1 t + \psi_3) + V_4 \exp(-\zeta_2 \beta_2 t) \cos(\omega_2 t + \psi_4)$$

where

$$\begin{aligned} V_3 &= \sqrt{(u_2 \omega_1 - u_1 \zeta_1 \beta_1)^2 + (u_1 \omega_1 + u_2 \zeta_1 \beta_1)^2} \\ V_4 &= \sqrt{(u_4 \omega_2 - u_3 \zeta_2 \beta_2)^2 + (u_3 \omega_2 + u_4 \zeta_2 \beta_2)^2} \\ \psi_3 &= \tan^{-1} \frac{u_1 \omega_1 + u_2 \zeta_1 \beta_1}{u_1 \zeta_1 \beta_1 - u_2 \omega_1} \\ \psi_4 &= \tan^{-1} \frac{u_3 \omega_2 + u_4 \zeta_2 \beta_2}{u_3 \zeta_2 \beta_2 - u_4 \omega_2} \end{aligned}$$

The formula for obtaining the constant r_i and u_i in equations (4.15a) and (4.15b) is outlined in Appendix C.

4.3 The variances and covariance of the Response

Since it is assumed that $\alpha(t)$ is Gaussian, and since the system is linear, it may be shown that $X_1(t)$ and $X_2(t)$ are also Gaussian^[2]. If the response process is Gaussian, then the mean values $\eta_1(t)$, $\eta_1(t)$, $\eta_2(t)$ and $\eta_2(t)$, and the variances characterize the process completely. It is therefore necessary to compute $\sigma_{X_1}^2$, $\sigma_{X_1}^2$, $\sigma_{X_1 X_1}^2$, $\sigma_{X_2}^2$, $\sigma_{X_2}^2$ and $\sigma_{X_2 X_2}^2$.

According to the definition, the variance $\sigma_{X_1}^2$ of a random process $X_1(t)$ can be expressed as

$$\sigma_{X_1}^2 = \langle [X_1(t) - \eta_1(t)]^2 \rangle \quad (4.17)$$

From equations (4.10a) and (4.13a), the variance $\sigma_{X_1}^2$ is given by

$$\begin{aligned} \sigma_{X_1}^2 &= \langle \int_0^t h_1(t-\tau_1) \alpha(\tau_1) d\tau_1 \int_0^t h_2(t-\tau_2) \alpha(\tau_2) d\tau_2 \rangle \\ &= \int_0^t \int_0^t h_1(t-\tau_1) h_2(t-\tau_2) \langle \alpha(\tau_1) \alpha(\tau_2) \rangle d\tau_1 d\tau_2 \end{aligned} \quad (4.18)$$

Since

$$\langle \alpha(\tau_1) \alpha(\tau_2) \rangle = \int_0^\infty S_\alpha \cos \omega(\tau_1 - \tau_2) d\omega \quad (4.19)$$

where S_α is the spectral density of $\alpha(t)$, the variance $\sigma_{X_1}^2$ is expressed as

$$\sigma_{X_1}^2 = 2D \int_0^\infty \int_0^t \int_0^t h_1(t-\tau_1) h_1(t-\tau_2) \cos(\tau_1 - \tau_2) d\tau_1 d\tau_2 d\omega \quad (4.20)$$

where S_α has been replaced by the white noise intensity $2D$.

Using the same procedure, the variance of $X_2(t)$ is

$$\sigma_{X_2}^2 = 2D \int_0^\infty \int_0^t \int_0^t h_2(t-\tau_1) h_2(t-\tau_2) \cos(\tau_1 - \tau_2) d\tau_1 d\tau_2 d\omega \quad (4.20b)$$

In order to obtain the variance $\sigma_{\dot{X}_1}^2$, the impulse response $h_3(t)$ of $\dot{X}_1(t)$ has to be calculated. By definition,

$$L[\dot{X}_1(t)] = sX_1(s) - X_1(0), \quad (4.21)$$

and further the initial condition for X_1 is given by $X_1 = \eta_1$ at $t=0$. Now equation (4.21) can be rewritten as

$$\begin{aligned} \dot{X}(t) &= L^{-1} [sX_1(s)] - X_1(0) \\ &= L^{-1} [sX_1(s)] - \eta_1(0) \\ &= \dot{\eta}_1(t) + L^{-1} [sH_1(s)\alpha(s)] \\ &= \dot{\eta}_1(t) + L^{-1} [H_3(s)\alpha(s)] \\ &= \dot{\eta}_1(t) + \int_0^t h_3(t-\tau)\alpha(\tau)d\tau \end{aligned} \quad (4.22)$$

where

$$H_3(s) = sH_1(s) \quad (4.23)$$

and

$$h_3(t) = L^{-1} [H_3(s)] \quad (4.24)$$

Once the impulse response $h_3(t)$ is known, the variance $\sigma_{\dot{X}_1}^2$ of $\dot{X}_1(t)$ can be computed in the following way.

The variance $\sigma_{\dot{X}_1}^2$ is defined as

$$\sigma_{\dot{X}_1}^2 = \langle [\dot{X}_1(t) - \dot{\eta}_1(t)]^2 \rangle \quad (4.25)$$

Substituting of equation (4.22) into equation (4.25), it gives as

$$\sigma_{\dot{X}_1}^2 = 2D \int_0^\infty \int_0^t \int_0^t h_3(t-\tau_1)h_3(t-\tau_2)\cos(\tau_1-\tau_2)d\tau_1d\tau_2d\omega \quad (4.26)$$

Since

$$h_4(t) = L^{-1} [H_4(s)] \quad (4.27)$$

where

$$H_4(s) = sH_3(s) \quad (4.28)$$

the variance $\sigma_{\dot{X}_2}^2$ of $\dot{X}_2(t)$ becomes

$$\sigma_{\dot{X}_2}^2 = 2D \int_0^\infty \int_0^t \int_0^t h_4(t-\tau_1) \cos \omega(\tau_1 - \tau_2) d\tau_1 d\tau_2 d\omega \quad (4.29)$$

The covariance $\sigma_{X_1 \dot{X}_1}^2$ of $X_1(t)$ and $\dot{X}_1(t)$ is defined as

$$\begin{aligned} \sigma_{X_1 \dot{X}_1}^2 &= \langle [X_1(t) - \bar{n}_1(t)] [\dot{X}_1(t) - \dot{\bar{n}}_1(t)] \rangle \\ &= \langle \int_0^t h_1(t-\tau_1) \alpha(\tau_1) d\tau_1 \int_0^t h_3(t-\tau_2) \alpha(\tau_2) d\tau_2 \rangle \\ &= 2D \int_0^\infty \int_0^t \int_0^t h_1(t-\tau_1) h_3(t-\tau_2) \cos \omega(\tau_1 - \tau_2) d\tau_1 d\tau_2 d\omega \end{aligned} \quad (4.30)$$

Similarly, the covariance $\sigma_{X_2 \dot{X}_2}^2$ of $X_2(t)$ and $\dot{X}_2(t)$ is expressed by

$$\sigma_{X_2 \dot{X}_2}^2 = 2D \int_0^\infty \int_0^t \int_0^t h_2(t-\tau_1) h_4(t-\tau_2) \cos \omega(\tau_1 - \tau_2) d\tau_1 d\tau_2 d\omega \quad (4.30a)$$

4.4 The first passage probability of $X_1(t)$ and $X_2(t)$

Since $X_1(t)$ and $X_2(t)$ are both Gaussian, and their ensemble mean, variance and covariance have been defined in the previous section, the probability density of $X_1(t)$ and $\dot{X}_1(t)$ can now be expressed as:

$$P(X_1, \dot{X}_1, t) = \frac{1}{2\pi\sigma_{X_1}\sigma_{\dot{X}_1}\sqrt{1-\rho_{X_1\dot{X}_1}^2}} \times \exp\left\{-\frac{1}{1-\rho_{X_1\dot{X}_1}^2} \left[\frac{(X_1-\eta_1)^2}{\sigma_{X_1}^2} + \frac{(\dot{X}_1-\dot{\eta}_1)^2}{\sigma_{\dot{X}_1}^2} - \frac{2\sigma_{X_1\dot{X}_1}}{\sigma_{X_1}^2\sigma_{\dot{X}_1}^2} (X_1-\eta_1)(\dot{X}_1-\dot{\eta}_1) \right]\right\} \quad (4.31)$$

Similarly, the probability density of $X_2(t)$ and $\dot{X}_2(t)$ can be written in the same way as equation (4.31), that is

$$P(X_2, \dot{X}_2, t) = \frac{1}{2\pi\sigma_{X_2}\sigma_{\dot{X}_2}\sqrt{1-\rho_{X_2\dot{X}_2}^2}} \times \exp\left\{-\frac{1}{1-\rho_{X_2\dot{X}_2}^2} \left[\frac{(X_2-\eta_2)^2}{\sigma_{X_2}^2} + \frac{(\dot{X}_2-\dot{\eta}_2)^2}{\sigma_{\dot{X}_2}^2} - \frac{2\sigma_{X_2\dot{X}_2}}{\sigma_{X_2}^2\sigma_{\dot{X}_2}^2} (X_2-\eta_2)(\dot{X}_2-\dot{\eta}_2) \right]\right\} \quad (4.32)$$

Since the probability densities shown in equations (4.31) and (4.32) have the same form as the probability density shown in equation (2.17), the first passage probability about a certain level y_A for $X_1(t)$ is

$$P_{f1} = 1 - \exp \left\{ - \int_0^t K_1 \left[\sigma_{A1}^2 + A_1 \sqrt{\pi/2} \sigma_{A1} - 2\sigma_{A1}^2 \int_0^{A_1/\sqrt{2}\sigma_{A1}} e^{-u^2} du \right. \right. \\ \left. \left. + \sqrt{2} A_1 \sigma_{A1}^2 \int_0^{A_1/\sqrt{2}\sigma_{A1}} e^{-u^2} du \right] dt \right\} \\ = 1 - \exp \left\{ - \int_0^t \left(K_1 \left[\sigma_{A1}^2 + A_1 \sqrt{\pi/2} \sigma_{A1} - 2\sigma_{A1}^2 g\left(\frac{A_1}{\sqrt{2}\sigma_{A1}}\right) \right. \right. \right. \quad (4.33) \\ \left. \left. \left. + \sqrt{2} A_1 \sigma_{A1}^2 h\left(\frac{A_1}{\sqrt{2}\sigma_{A1}}\right) \right] dt \right) \right\}$$

where

$$K_1 = \frac{\exp \left\{ -(y_A - \eta_1)^2 / 2\sigma_{x1}^2 \right\}}{2\pi\sigma_{x1}\sigma_{\dot{x}1} \sqrt{1 - \rho_{x1\dot{x}1}^2}}$$

$$A_1 = \eta_1 + \sigma_{x1\dot{x}1}^2 / \sigma_{\dot{x}1}^2 (y_A - \eta_1) \quad (4.34)$$

$$\sigma_{A1}^2 = (1 - \rho_{x1\dot{x}1}^2) \sigma_{x1\dot{x}1}^2$$

$$y_A = k_1 y_s$$

Also, the first passage probability about a certain level y_B for $x_2(t)$ is

$$P_{f2} = 1 - \exp \left\{ - \int_0^t K_2 \left[\sigma_{B1}^2 + B_1 \sqrt{\pi/2} \sigma_{B1} - 2\sigma_{B1}^2 g\left(\frac{B_1}{\sqrt{2}\sigma_{B1}}\right) \right. \right. \\ \left. \left. + \sqrt{2} B_1 \sigma_{B1}^2 h\left(\frac{B_1}{\sqrt{2}\sigma_{B1}}\right) \right] dt \right\} \quad (4.35)$$

where

$$K_2 = \frac{\exp[-(y_B - \eta_2)^2 / 2\sigma_{X_2}^2]}{2\pi\sigma_{X_2}\sigma_{\dot{X}_2}\sqrt{1-\rho_{X_2\dot{X}_2}^2}}$$

$$B_1 = \dot{\eta}_2 + \frac{\sigma_{X_2\dot{X}_2}^2}{\sigma_{X_2}^2} (y_B - \eta_2)$$

(4.36)

$$\sigma_{B_1}^2 = (1 - \rho_{X_2\dot{X}_2}^2) \sigma_{X_2\dot{X}_2}^2$$

$$y_B = k_2 y_S$$

4.5 The first passage probability for steady state response

The steady state is defined as the settling time state at which time t is large compared to t_s . In other words, equations (4.15a) and (4.15b) show that as time t approaches infinity, the mean values η_1 and η_2 become zero. That is,

$$\langle X_1 \rangle_s = \eta_{1s} = \eta_1|_{t \rightarrow \infty} = 0$$

$$\langle X_2 \rangle_s = \eta_{2s} = \eta_2|_{t \rightarrow \infty} = 0$$

(4.37)

Similary,

$$\langle \dot{X}_1 \rangle_s = \eta_{1s} = \dot{\eta}_1|_{t \rightarrow \infty} = 0$$

$$\langle \dot{X}_2 \rangle_s = \eta_{2s} = \dot{\eta}_2|_{t \rightarrow \infty} = 0$$

(4.38)

The variance $\sigma_{X_1}^2$ can be expressed as

$$(\sigma_{X_1}^2)_s = \langle [X_1(t) - \eta_1]^2 \rangle|_{t \rightarrow \infty}$$

$$= \langle X_1(t) \rangle|_{t \rightarrow \infty}$$

$$= \int_0^\infty S_\alpha |H_1(j\omega)|^2 d\omega$$

$$= D \int_0^\infty |H_1(j\omega)|^2 d\omega$$

(4.39)

The magnitude $|H_1(j\omega)|^2$ can be calculated by substituting $s=j\omega$ in equation (4.7a), that is,

$$\begin{aligned} |H_1(j\omega)|^2 &= \left| \frac{-\omega^2 + j\omega d_1 + d_2}{\omega^4 - a_1 j\omega^3 - a_2 \omega^2 + a_3 j\omega + a_4} \right|^2 \\ &= \frac{\omega^4 + \theta_1 \omega^2 + \theta_2}{\omega^8 + \theta_3 \omega^6 + \theta_4 \omega^4 + \theta_5 \omega^2 + \theta_6} \end{aligned} \quad (4.40)$$

where

$$\begin{aligned} \theta_1 &= 2d_2 + d_1 \\ \theta_2 &= d_2 \\ \theta_3 &= a_1 - 2a_2 \\ \theta_4 &= a_2 + 2a_4 - 2a_1 a_3 \\ \theta_5 &= a_3 - 2a_2 a_4 \\ \theta_6 &= a_4 \end{aligned}$$

The variance $\sigma_{\dot{X}_1}^2$ during steady state motion is given by

$$\begin{aligned} (\sigma_{\dot{X}_1})_s^2 &= \langle [\dot{X}_1(t) - \eta_1]^2 \rangle |_{t \rightarrow \infty} \\ &= \langle \dot{X}_1(t)^2 \rangle |_{t \rightarrow \infty} \\ &= \int_0^\infty S_\alpha |\omega H_1(j\omega)|^2 d\omega \\ &= D \int_{-\infty}^\infty \frac{\omega^6 + \theta_1 \omega^4 + \theta_2 \omega^2}{\omega^8 + \theta_3 \omega^6 + \theta_4 \omega^4 + \theta_5 \omega^2 + \theta_6} d\omega \end{aligned} \quad (4.41)$$

The covariance $\sigma_{X_1 \dot{X}_1}$ at steady state is equal to zero, that is,

$$\begin{aligned} (\sigma_{X_1 \dot{X}_1})_s^2 &= \langle [X_1(t) - \eta_1][\dot{X}_1(t) - \eta_1] \rangle |_{t \rightarrow \infty} \\ &= \langle X_1(t) \dot{X}_1(t) \rangle |_{t \rightarrow \infty} \\ &= \int_0^\infty \omega S_\alpha \sin \omega \tau |H_1(j\omega)|^2 d\omega |_{\tau=0} \\ &= 0 \end{aligned} \quad (4.42)$$

Using a similar procedure for the response $X_2(t)$, $\sigma_{X_2}^2$, $\sigma_{\dot{X}_2}^2$ and $\sigma_{X_2\dot{X}_2}^2$ at steady state condition, it may be obtained as,

$$\begin{aligned} (\sigma_{X_2})_s^2 &= \int_0^\infty S_\alpha |H_2(j\omega)|^2 d\omega \\ &= D \int_0^\infty \frac{d_1^2 \omega^2 + d_2^2}{\omega^8 + \theta_3 \omega^6 + \theta_4 \omega^4 + \theta_5 \omega^2 + \theta_6} d\omega \end{aligned} \quad (4.43)$$

Also,

$$(\sigma_{\dot{X}_2})^2 = D \int_0^\infty \frac{d_1^2 \omega^4 + d_2^2}{\omega^8 + \theta_3 \omega^6 + \theta_4 \omega^4 + \theta_5 \omega^2 + \theta_6} d\omega \quad (4.44)$$

and

$$(\sigma_{X_2\dot{X}_2})_s^2 = 0 \quad (4.45)$$

In order to calculate those integrals in equations (4.41) through (4.44), the method residues is used, which states

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{P(X)}{Q(X)} dX \\ &= 2\pi j \sum \text{residue of } \frac{P(X)}{Q(X)} \text{ at the poles in the} \\ &\text{upper half plane.} \end{aligned}$$

To obtain the poles of the polynomial in the denominator in equation (4.44), the Newton Raphson method is applied.

After evaluating the integrals, the variances $(\sigma_{X_1})_s^2$, $(\sigma_{\dot{X}_1})_s^2$, $(\sigma_{X_2})_s^2$ and $(\sigma_{\dot{X}_2})_s^2$ are obtained. A computer program was developed to determine the variances of X_1 . Figures 4.2 and 4.3 show the variances $(\sigma_{X_1})_s^2$ and $(\sigma_{X_2})_s^2$ against the natural frequency ratio ω_{n1}/ω_{n2} respectively by assuming the mass $m_2 = 0.2m_1$, $\omega_{n2} = 1.0$ rad/Sec and the damping ratios ζ_1 and ζ_2 are equal.

From this, it may be noticed that as the damping ratio or the frequency ratio becomes smaller, both the variances σ_{x_1} and σ_{x_2} are larger. As the variances of the system are well defined, using the same procedure as in the single-degree-of-freedom system, the probability densities $p(x_1, \dot{x}_1)$ and $p(x_2, \dot{x}_2)$ are found to be of the form,

$$p(x_1, \dot{x}_1) = \frac{1}{2\pi (\sigma_{x_1})_s (\sigma_{\dot{x}_1})_s} \exp \left[-\frac{x_1^2}{2(\sigma_{x_1})_s^2} - \frac{\dot{x}_1^2}{2(\sigma_{\dot{x}_1})_s^2} \right] \quad (4.46a)$$

$$p(x_2, \dot{x}_2) = \frac{1}{2\pi (\sigma_{x_2})_s (\sigma_{\dot{x}_2})_s} \exp \left[-\frac{x_2^2}{2(\sigma_{x_2})_s^2} - \frac{\dot{x}_2^2}{2(\sigma_{\dot{x}_2})_s^2} \right] \quad (4.46b)$$

The above probability densities have the same pattern as the probability density for single-degree-of-freedom system.

Thus the first passage probability of x_1 and x_2 about certain levels y_A and y_B can be expressed respectively as,

$$(P_{f_1})_s = 1 - \exp \left[-\frac{(\sigma_{\dot{x}_1})_s}{2\pi (\sigma_{x_1})_s} T e^{-\frac{y_A^2}{2(\sigma_{x_1})_s^2}} \right] \quad (4.47a)$$

$$(P_{f_2})_s = 1 - \exp \left[-\frac{(\sigma_{\dot{x}_2})_s}{2\pi (\sigma_{x_2})_s} T e^{-\frac{y_B^2}{2(\sigma_{x_2})_s^2}} \right] \quad (4.47b)$$

By defining

$$\begin{aligned} N_A &= \frac{(\sigma_{\dot{x}_1})_s}{2\pi (\sigma_{x_1})_s} e^{-\frac{y_A^2}{2(\sigma_{x_1})_s^2}} \\ &= \frac{(\sigma_{\dot{x}_1})_s}{2\pi (\sigma_{x_1})_s} e^{-(K_A^2/2)} \end{aligned}$$

and

$$N_B = \frac{(\sigma_{\dot{x}_2})_s}{2 (\sigma_{x_2})_s} e^{-[Y_B^2/2 (\sigma_{x_2})_s^2]} \\ = \frac{(\sigma_{\dot{x}_2})_s}{2 (\sigma_{x_2})_s} e^{-(K_B^2/2)}$$

where

$$K_A = Y_A/(\sigma_{x_1})_s$$

and

$$K_B = Y_B/(\sigma_{x_2})_s$$

Equations (4.47a) and (4.47b) can be simplified as,

$$(P_{f1})_s = 1 - \exp(-N_A T) \quad (4.48a)$$

and

$$(P_{f2})_s = 1 - \exp(-N_B T) \quad (4.48b)$$

Figures 4.6 and 4.7 give the curves of the first passage probabilities $(P_{f1})_s$ and $(P_{f2})_s$ against the multiple of natural frequency ω_{n1} by assuming $\omega_{n2} = 1.0$ rad/sec and $m_2 = 0.2m_1$ and $\zeta_1 = \zeta_2$ for different values of K_A and K_B .

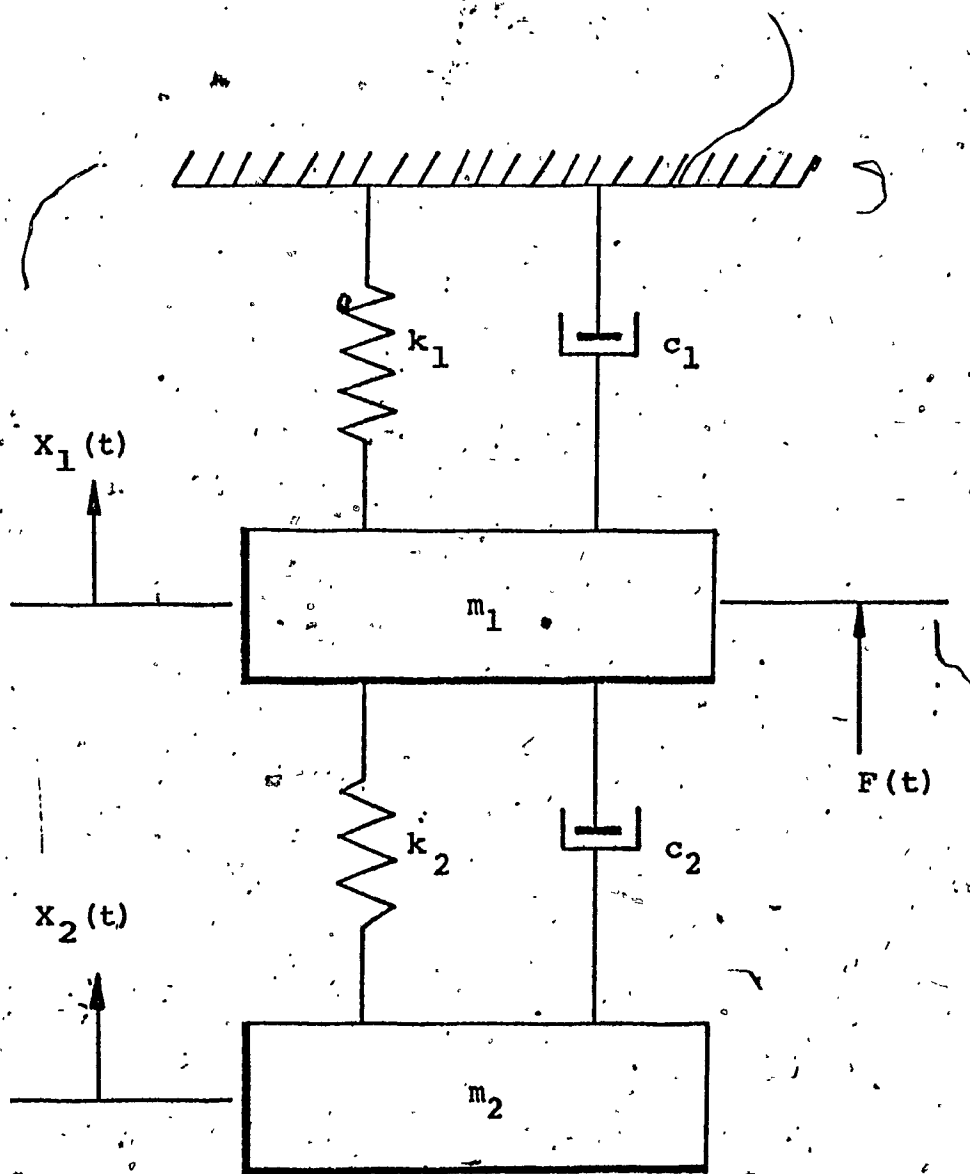


Figure 4.1 Two-degree-of-freedom system

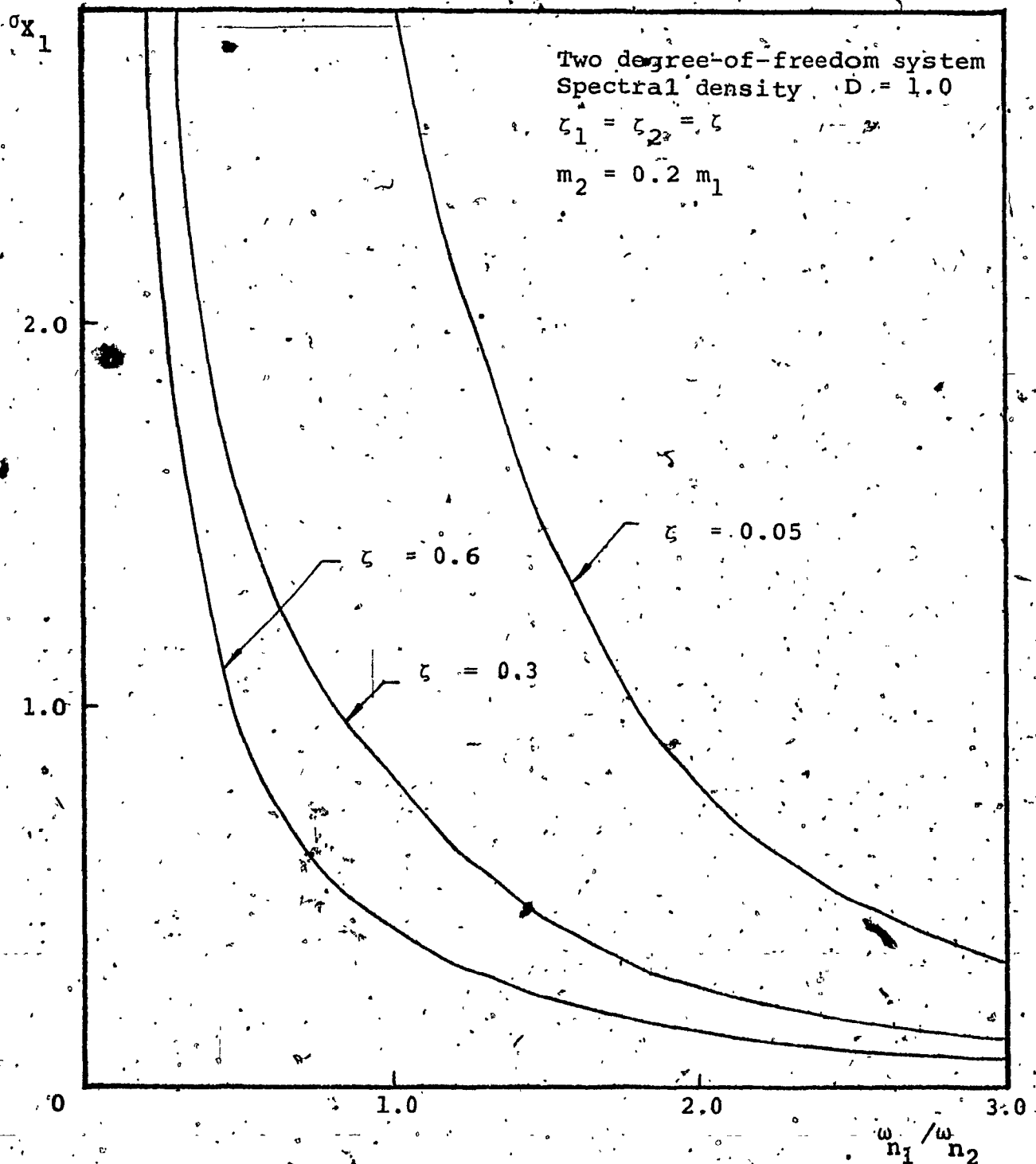


Figure 4.2 Variance of X_1 against the natural frequency ratio at steady state condition

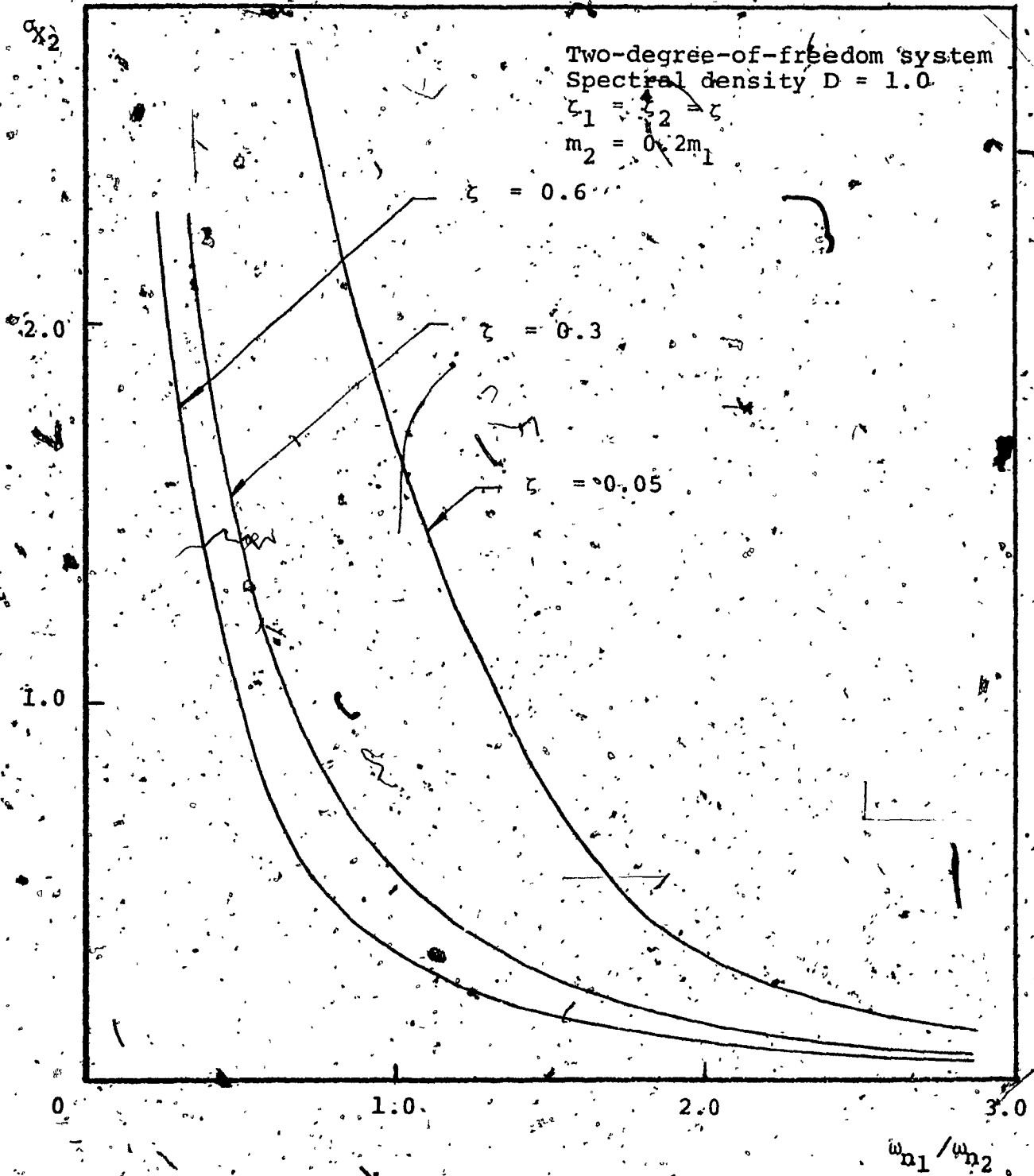


Figure 4.3 Variance of X_2 against the natural frequency ratio at steady state condition

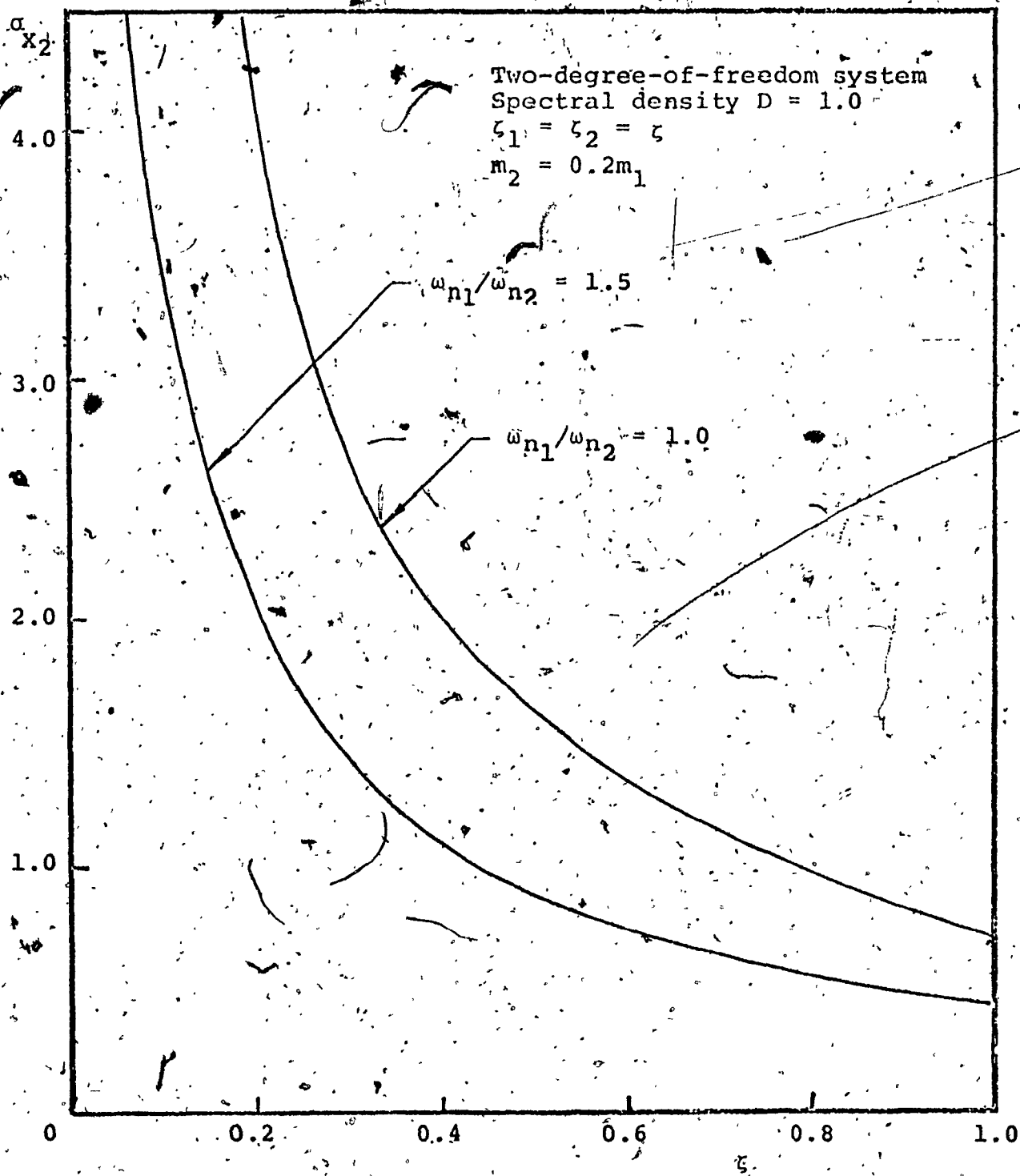


Figure 4.4 Variance of x_2 against damping ratio at steady state condition

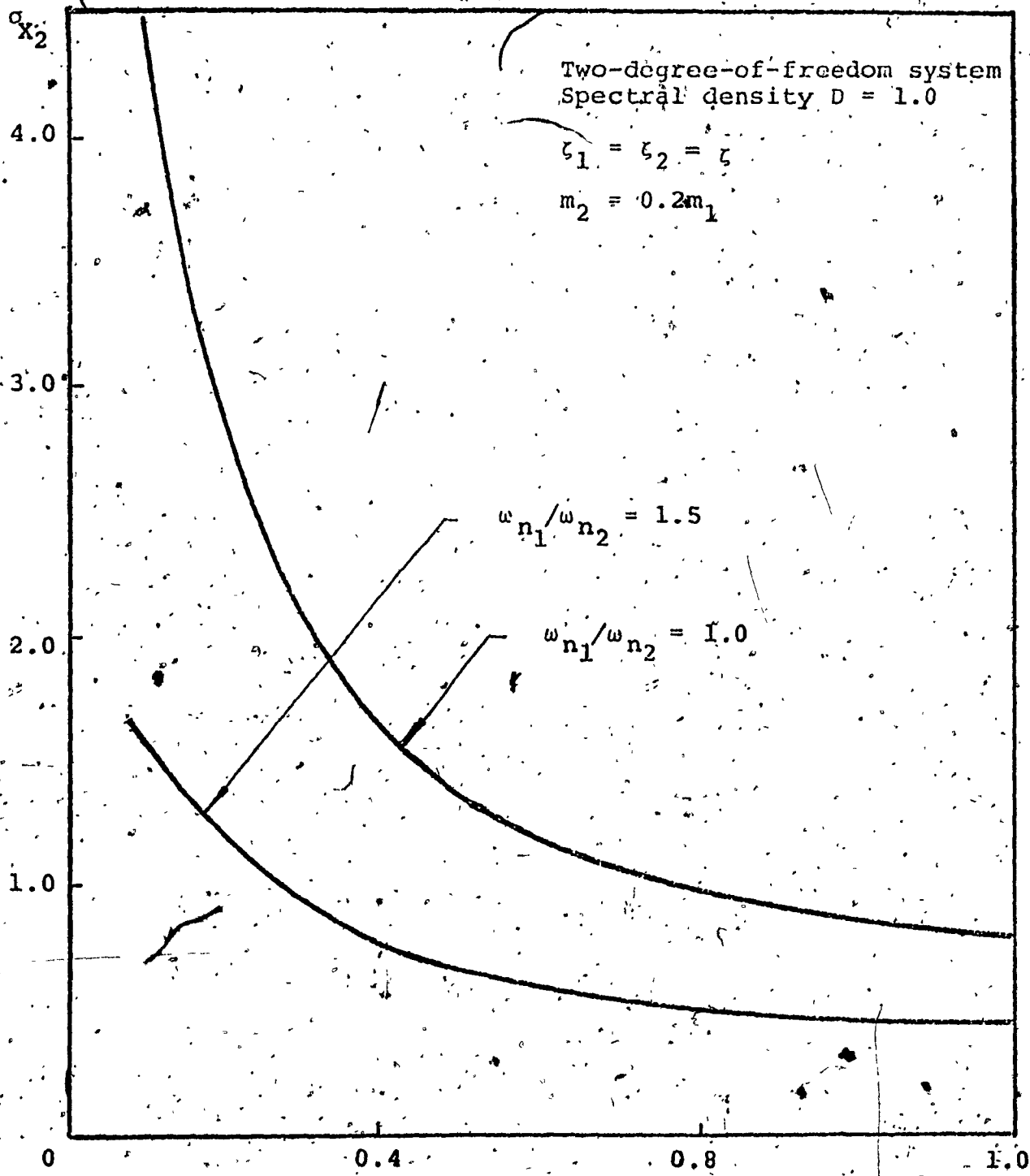


Figure 4.5 Variance of X_2 against damping ratio at steady state condition

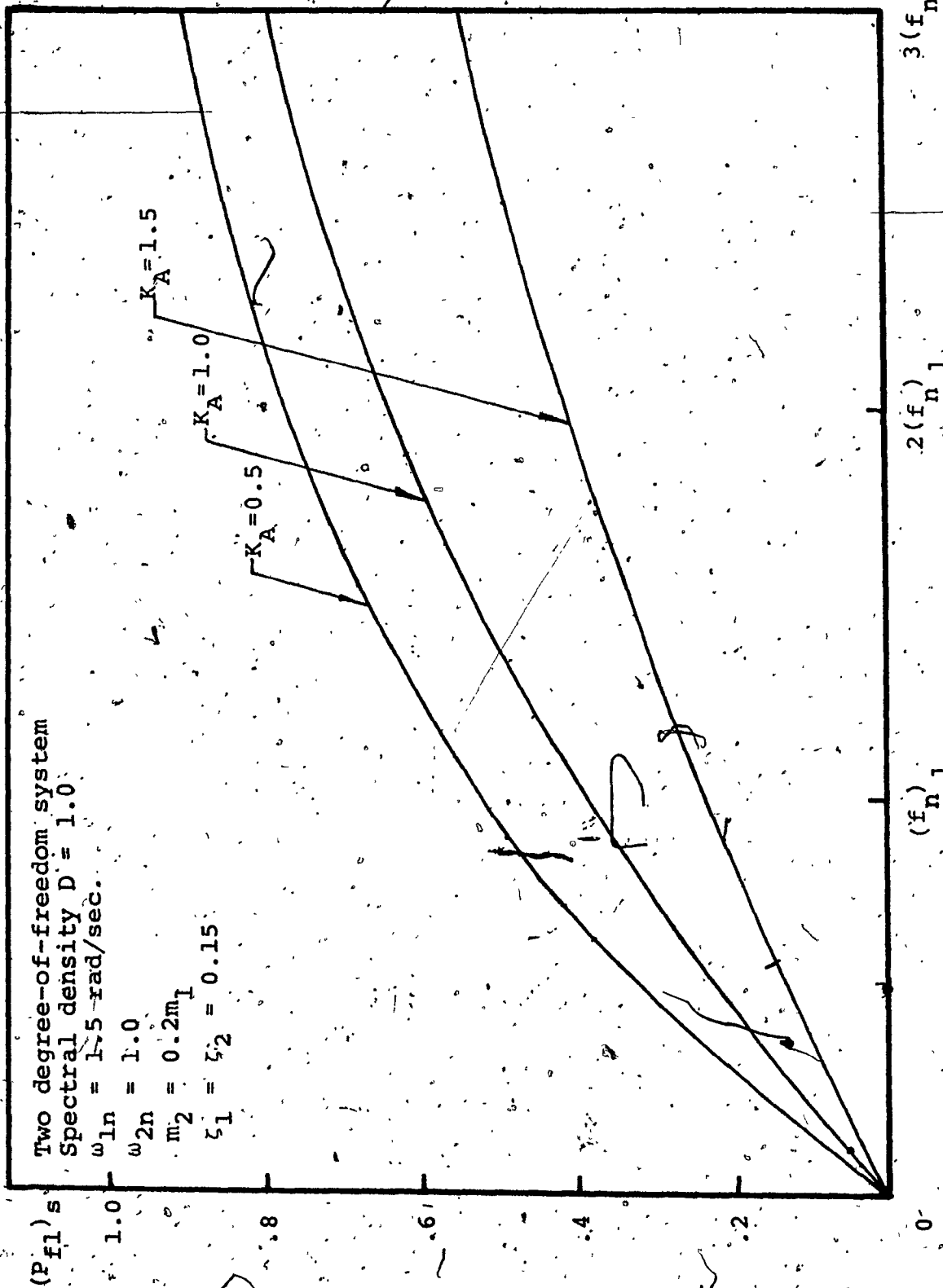


Figure 4.6 Steady-state first passage probability of response x_1 against multiple of natural frequency

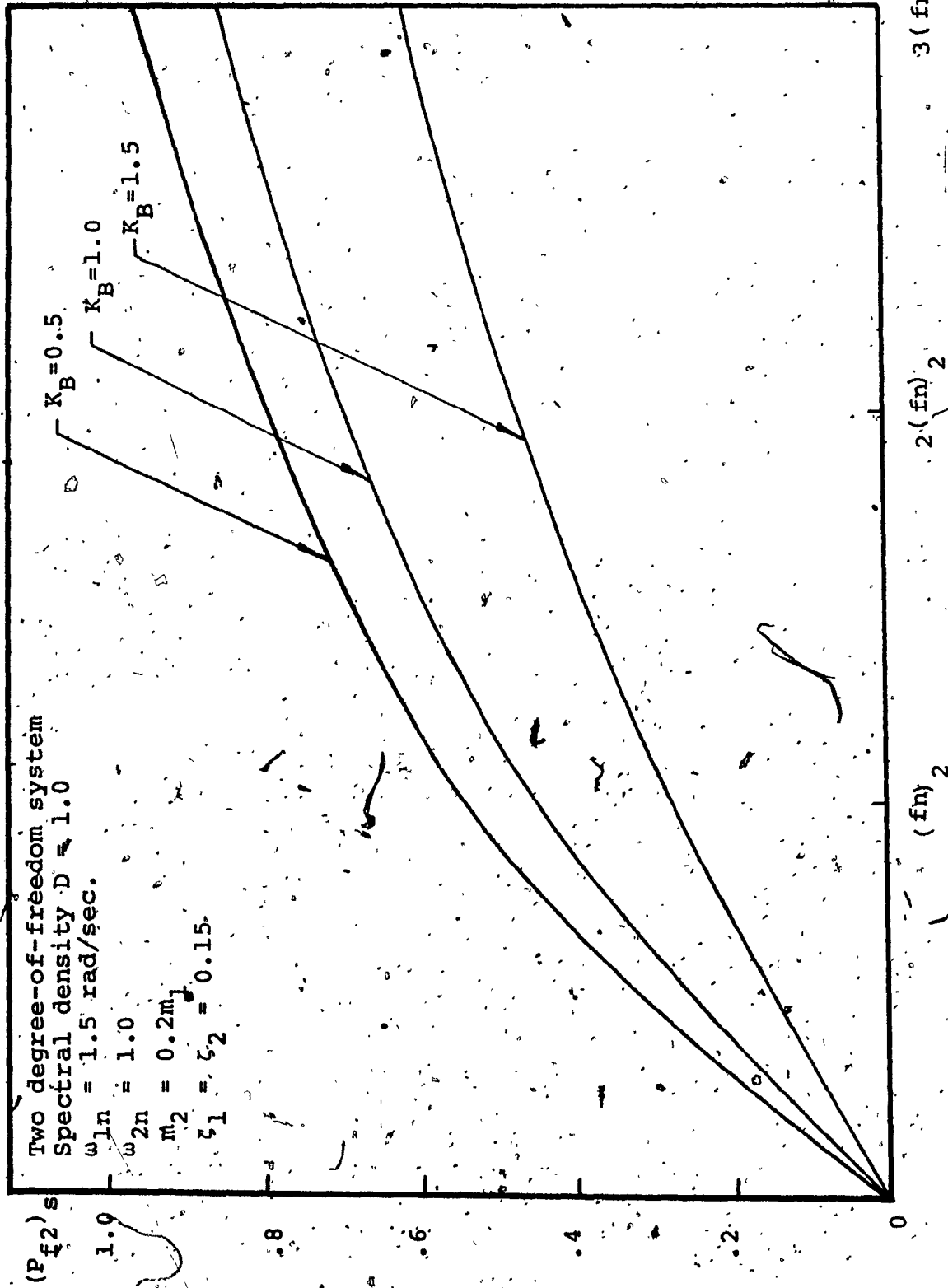


Figure 4.7 Steady state first passage probability of response X_2 against multiple of natural frequency f_2

CHAPTER FIVE

CONCLUDING REMARKS

In the previous chapters, detailed derivations of the first passage time probability for simple mechanical systems subjected to stochastic excitation were presented. The analysis took into account both transient and steady state motion of the system. The first passage probability was derived using the complete solution of the Fokker-Planck equation giving the time dependent probability of the response.

The results of the investigation are presented in the Figures 3.5 and 3.6. In the Figure 3.5, the transient first passage probability is shown against multiples of natural frequency for different safe levels and given initial conditions. From the curves, it is seen that the first passage probability decreases with an increase of the safe level. On the other hand, for a given safe level, the transient first passage probability decreases with increasing damping ratio. The damping ratios considered are $\zeta = 0.0125$ and $\zeta = 0.015$, which are the damping ratios that actually exist in machines and structures. It is also apparent from the Figure that the first passage probability depends on initial conditions, although Figure 3.5 has been plotted for one initial condition. In Figure 3.6, the first passage probability is shown against the settling time for different values of initial conditions and different values of natural frequency for a given safe level. It is seen from the Figure that the first passage probability decreases with increasing damping ratio for a given initial condition and a given natural frequency. The Figure also shows

the effect of the natural frequency of the system on the first passage probability. If the natural frequency decreases, and keeping other variables constant, the first passage probability decreases. That is, the system with low natural frequency has higher first passage probability. The most important conclusion that can be drawn from Figure 3.6 is that when the system has low damping, the first passage of failure occurs during the transient motion itself. This fact shows the importance of the study undertaken in this investigation. A detailed study of this phenomenon, taking into account the initial conditions, natural frequency, safe level and settling time for low damping of the order 0.01 to 0.02 must be initiated, because this would be in the region of engineering interest.

The results of the investigation of the two-degree-of-freedom linear system are presented in Figures 4.6 and 4.7.

In Figure 4.6, the steady state first passage probability of response $X_1(t)$ against multiples of natural frequency is plotted for different values of the constant K_A for a given damping ratio and a mass ratio. It is seen from the Figure that the first passage probability of the response of first mass decreases with the increase of K_A . Since K_A is defined as the ratio of safe limit to variance, increasing K_A implies either increasing the safe limit or decreasing the variance. The effect of variable damping, variable mass ratio and variable natural frequencies must be investigated. From Figure 4.7, the first passage probability of response of mass m_2 is shown against multiples

of natural frequency. The same conclusion holds good in this case. For a two-degree-of-freedom system, the steady state first passage probability increases with increasing of the multiples of natural frequency of the system. Systems with low frequency have lower values of steady state first passage probability, while systems with high frequency have higher first passage probability. An investigation of the transient first passage probability for a two-degree-of-freedom system is complicated because of the difficulties of obtaining these stochastic parameters, but this problem will be taken into consideration in the future.

The first passage probability can be considered as the probability of failure of any system. The lower the value of this probability distribution, the higher the reliability of the system. It is then necessary to evaluate this probability for randomly excited mechanical systems in order to determine the durability and performance of the system during its operating life. If the system possesses a large damping, then the first passage probability will be governed by the stationary Fokker-Planck equation. But for the system with a low damping, the failure probability can only be determined from the transient solution of the stochastic differential equation describing the motion of the system. The theoretical analysis developed in this thesis is valid for steady as well as transient state of the system.

The investigation presented here has many applications.

It is known that many mechanical systems in reality are subjected to randomly fluctuating forces and the reliability of the system will largely depend on the first passage time probability. The vibration signature of rotating and reciprocating machinery is essentially a random signal and hence any maintenance schedule based upon monitored data must be formulated, from statistical information, such as first passage probability. The procedure is also equally applicable for determining surface roughness characteristics, fatigue failure criteria for randomly stressed structures, etc.

The theoretical analysis presented here may be extended to cover higher degree-of-freedom systems, but the mathematical difficulties involved would be large. The excitation of the system has been assumed to be of the white noise type but this may not be the case in reality. In certain cases, the system may be subjected to forces which may not have a wide band frequency and hence cannot be considered as white noise. For such special cases, the first passage probability can be computed using a procedure similar to one presented in this thesis. Another interesting concept that can be derived from this investigation is the probability of the duration of the excursion. The reliability of a system can then be based on both the probability of first passage and the probability of the duration of the signal above a particular level. These two probabilities can then be used to indicate the total reliability of the system.

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APPENDIX A

Solution of the symmetrical partial differential equation of the form

$$\frac{\partial p}{\partial t} = b \frac{\partial (Z_1 p)}{\partial Z_1} + a \frac{\partial (Z_2 p)}{\partial Z_2} + D \left(\frac{\partial}{\partial Z_1} + \frac{\partial}{\partial Z_2} \right)^2 p \quad (A.1)$$

is discussed in this appendix.

In order to solve the equation shown above, the Fourier transform technique is employed. The Fourier transform of $p(Z_1, Z_2, t)$ is defined by

$$f(\zeta_1, \zeta_2; t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(Z_1, Z_2; t) \exp(i\zeta_1 Z_1 + i\zeta_2 Z_2) dZ_1 dZ_2 \quad (A.2)$$

By multiplying equation (A.1) by a factor $\exp(i\zeta_1 Z_1 + i\zeta_2 Z_2)$ and integrating in the regions $-\infty \leq Z_1 \leq \infty$ and $-\infty \leq Z_2 \leq \infty$, equation (A.1) is transformed into a new symmetrical differential equation in terms of variables ζ_1, ζ_2 and t .

$$\frac{\partial f}{\partial t} = -b\zeta_1 \frac{\partial f}{\partial \zeta_1} - a\zeta_2 \frac{\partial f}{\partial \zeta_2} - D(\zeta_1^2 + 2\zeta_1\zeta_2 + \zeta_2^2)f \quad (A.3)$$

Equation (A.3) can be expressed as

$$\frac{\partial f}{\partial t} = \lambda_1 \zeta_1 \frac{\partial f}{\partial \zeta_1} + \lambda_2 \zeta_2 \frac{\partial f}{\partial \zeta_2} + \frac{1}{2}(\sigma_1 \zeta_1^2 + 2\sigma_{12} \zeta_1 \zeta_2 + \sigma_2 \zeta_2^2) f \quad (A.4)$$

where

$$\lambda_1 = -b$$

$$\lambda_2 = -a$$

$$\begin{aligned}\sigma_1 &= -2D \\ \sigma_{12} &= -D \\ \sigma_2 &= -2D\end{aligned}\quad (A.5)$$

The above equation is a linear first order partial differential equation, and the solution is of the form

$$f(\zeta_1, \zeta_2, t) = \phi(C_1, C_2) g(\zeta_1, \zeta_2) \quad (A.6)$$

The variables C_1 and C_2 are obtained from the following subsidiary equation

$$dt = \frac{d\zeta_1}{\lambda_1 \zeta_1} = \frac{d\zeta_2}{\lambda_2 \zeta_2} = \frac{df}{\frac{1}{2}f(\sigma_1 \zeta_1^2 + 2\sigma_{12} \zeta_1 \zeta_2 + \sigma_2 \zeta_2^2)} \quad (A.7)$$

From the first two relations of equation (A.7)

$$dt = \frac{d\zeta_1}{\lambda_1 \zeta_1} \quad (A.8)$$

The solution of equation (A.8) is

$$C_1 = \zeta_1 e^{\lambda_1 t} \quad (A.9)$$

Similarly, the solution of the differential equation

$$dt = \frac{d\zeta_2}{\lambda_2 \zeta_2}$$

is

$$C_2 = \zeta_2 e^{+\lambda_2 t} \quad (A.10)$$

Since

$$\frac{d\zeta_1}{\lambda_1 \zeta_1} = \frac{d\zeta_2}{\lambda_2 \zeta_2} \quad (A.11)$$

the above equation can be rewritten as

$$\frac{\zeta_2 d\zeta_1 + \zeta_1 d\zeta_2}{\lambda_1 \zeta_1 \zeta_2 + \lambda_2 \zeta_1 \zeta_2} = \frac{\zeta_2 d\zeta_1}{\lambda_1 \zeta_1 \zeta_2} \quad (A.12)$$

or

$$\frac{d(\zeta_1 \zeta_2)}{(\lambda_1 + \lambda_2)} = \frac{\zeta_2 d\zeta_1}{\lambda_1}$$

From equation (A.7),

$$\frac{d\zeta_1}{\lambda_1 \zeta_1} = \frac{df}{\frac{1}{2}f(\sigma_1 \zeta_1^2 + 2\sigma_{12} \zeta_1 \zeta_2 + \sigma_2 \zeta_2^2)} \quad (A.12)$$

or

$$\left(\frac{\sigma_1 \zeta_1}{\lambda_1} + \frac{\sigma_{12} \zeta_2}{\lambda_1} + \frac{\sigma_2 \zeta_2^2}{\zeta_1 \lambda_1} \right) d\zeta_1 = \frac{df}{f}$$

Substituting equations (A.11) and (A.12) into the above equation (A.12)

$$\frac{\sigma_1}{2\lambda_1} d\zeta_1 + \frac{\sigma_{12}}{\lambda_1 + \lambda_2} d(\zeta_1 \zeta_2) + \frac{\sigma_2}{2\lambda_2} \zeta_2 d\zeta_2 = \frac{df}{f} \quad (A.13)$$

The solution of the equation (A.13) is

$$\frac{\sigma_1}{4\lambda_1} \zeta_1^2 + \frac{\sigma_{12}}{\lambda_1 + \lambda_2} \zeta_1 \zeta_2 + \frac{\sigma_2}{4\lambda_2} \zeta_2^2 = \ln f$$

or

$$g(\zeta_1, \zeta_2) = \exp \left[\frac{\sigma_1}{4\lambda_1} \zeta_1^2 + \frac{\sigma_{12}}{\lambda_1 + \lambda_2} \zeta_1 \zeta_2 + \frac{\sigma_2}{4\lambda_2} \zeta_2^2 \right] \quad (A.14)$$

Therefore, the complete solution of equation (A.5) is

$$f(\zeta_1, \zeta_2, t) = \phi(\zeta_1 e^{+\lambda_1 t}, \zeta_2 e^{+\lambda_2 t}) \exp \left[\frac{\sigma_1}{4\lambda_1} \zeta_1^2 + \frac{\sigma_{12}}{\lambda_1 + \lambda_2} \zeta_1 \zeta_2 + \frac{\sigma_2}{4\lambda_2} \zeta_2^2 \right] \quad (A.15)$$

Initial conditions are employed for determining the function $\phi(\zeta_1 e^{-\lambda_1 t}, \zeta_2 e^{-\lambda_2 t})$ in equation (A.15). At $t = 0$, it can be seen from equation (A.2) that

$$f(\zeta_1, \zeta_2, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(z_1, z_2, 0) dz_1 dz_2 e^{i\zeta_1 z_{10} + i\zeta_2 z_{20}} = e^{i\zeta_1 z_{10} + i\zeta_2 z_{20}} \quad (A.16)$$

where z_{10} and z_{20} are the values of z_1 and z_2 at time $t = 0$.

Then, equation (A.15) can be written for $t = 0$, as

$$f(\zeta_1, \zeta_2, 0) = \phi(\zeta_1 e^{\lambda_1 t}, \zeta_2 e^{\lambda_2 t}) \Big|_{t=0} \exp\left[\frac{\sigma_1}{4\lambda_1} \zeta_1^2 + \frac{\sigma_2}{4\lambda_2} \zeta_2^2 + \frac{\sigma_{12}}{\lambda_1 + \lambda_2} \zeta_1 \zeta_2\right]$$

Hence,

$$\phi(\zeta_1, \zeta_2) \Big|_{t=0} = \exp\left\{-\left[\frac{\sigma_1}{4\lambda_1} \zeta_1^2 + \frac{\sigma_{12}}{\lambda_1 + \lambda_2} \zeta_1 \zeta_2 + \frac{\sigma_2}{4\lambda_2} \zeta_2^2\right]\right\} \times e^{i\zeta_1 z_{10} + i\zeta_2 z_{20}}$$

or

$$\phi(\zeta_1 e^{\lambda_1 t}, \zeta_2 e^{\lambda_2 t}) = \exp\left\{-\left[\frac{\sigma_1}{4\lambda_1} \zeta_1^2 e^{2\lambda_1 t} + \frac{\sigma_{12}}{\lambda_1 + \lambda_2} \zeta_1 \zeta_2 e^{(\lambda_1 + \lambda_2)t} + \frac{\sigma_2}{4\lambda_2} \zeta_2^2 e^{2\lambda_2 t}\right]\right\} \times e^{i\zeta_1 z_{10} e^{\lambda_1 t} + i\zeta_2 z_{20} e^{\lambda_2 t}} \quad (A.17)$$

The complete solution of $f(\zeta_1, \zeta_2, t)$ is now written as

$$f(\zeta_1, \zeta_2, t) = \exp\{i(\zeta_1 z_{10} e^{\lambda_1 t} + \zeta_2 z_{20} e^{\lambda_2 t}) + \frac{\sigma_1 \zeta_1^2}{4\lambda_1} [1 - \exp(2\lambda_1 t)] + \frac{\sigma_{12} \zeta_1 \zeta_2}{2(\lambda_1 + \lambda_2)} [1 - \exp((\lambda_1 + \lambda_2)t)] + \frac{\sigma_2 \zeta_2^2}{4\lambda_2} [1 - \exp(2\lambda_2 t)]\} \quad (A.18)$$

In order to transform equation (A.18) back in terms of the probability density $p(z_1, z_2, t)$, the Fourier transform of a Gaussian distribution density is employed and is explained

as follows.

The one dimension Gaussian distribution probability density $p(X)$ can be written as

$$p(X) = \frac{1}{\sqrt{2\pi}\sigma} \exp[-(X-\bar{X})^2/2\sigma^2] \quad (A.19)$$

where σ is the variance of variable X and \bar{X} is the mean value of X .

The Fourier transform of equation (A.19) is

$$\begin{aligned} \phi(v) &= \int_{-\infty}^{\infty} p(X) \exp(ivX) dX \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{[ivX - (X-\bar{X})^2/2\sigma^2]} dX \end{aligned} \quad (A.20)$$

The argument in the exponential term under the integral in equation (A.20) may be rearranged as

$$\begin{aligned} ivX - \frac{(X-\bar{X})^2}{2\sigma^2} &= \frac{1}{2} \{ 2ivX + [i \frac{(X-\bar{X})}{\sigma}]^2 \} \\ &= \frac{1}{2} \left[\left(\frac{X-\bar{X}}{\sigma} - iv\sigma \right)^2 + iv\bar{X} - \frac{v^2\sigma^2}{2} \right] \end{aligned} \quad (A.21)$$

Substituting equation (A.21) back into equation (A.20),

$$\begin{aligned} \phi(v) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left[\left(\frac{X-\bar{X}}{\sigma} - iv\sigma \right)^2 + iv\bar{X} - \frac{v^2\sigma^2}{2} \right]} dX \\ &= e^{iv\bar{X} - v^2\sigma^2/2} \times \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left(\frac{X-\bar{X}}{\sigma} - iv\sigma \right)^2} dX \end{aligned} \quad (A.22)$$

Suppose

$$\gamma = \left(\frac{x - \bar{x}}{\sigma} - i\nu\sigma \right)$$

Then, the integral in (A.22) can be written as

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left[\frac{x - \bar{x}}{\sigma} - i\nu\sigma \right]^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty - i\nu\sigma}^{\infty + i\nu\sigma} e^{-\frac{\gamma^2}{2}} d\gamma = 1$$

Therefore, the Fourier transform of the one-dimensional Gaussian density distribution $p(x_1)$ is expressed as

$$\phi(\nu) = e^{i\nu\bar{x} - \nu^2\sigma^2/2} \quad (A.23)$$

The Fourier transform of a two-dimensional independent Gaussian distribution density is expressed as

$$\begin{aligned} \phi(\nu_1, \nu_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{(x_1 - \bar{x}_1)^2}{2\sigma_1^2} - \frac{(x_2 - \bar{x}_2)^2}{2\sigma_2^2}} \\ &\quad \times e^{i\nu_1 x_1 + i\nu_2 x_2} dx_1 dx_2 \\ &= e^{[i\nu_1 \bar{x}_1 + i\nu_2 \bar{x}_2 - \nu_1^2 \sigma_1^2 / 2 - \nu_2^2 \sigma_2^2 / 2]} \end{aligned} \quad (A.24)$$

The matrix form for equation (A.24) is

$$\phi(\nu_1, \nu_2) = \exp[i\bar{x}^T \nu - \frac{1}{2} \nu^T \Sigma \nu] \quad (A.25)$$

where

$$X = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}$$

$$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

The Fourier transform of a two-dimensional Gaussian joint probability density may be developed as follows:
The probability density in this case is written as

$$p(X_1, X_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{X_1X_2}^2}} \exp \left\{ -\frac{1}{2(1-\rho_{X_1X_2}^2)} \left[\frac{(X_1 - \bar{X}_1)^2}{\sigma_1^2} + \frac{(X_2 - \bar{X}_2)^2}{\sigma_2^2} - \frac{2\rho_{X_1X_2}}{\sigma_1\sigma_2} (X_1 - \bar{X}_1)(X_2 - \bar{X}_2) \right] \right\} \quad (A.26)$$

The above two dependent random variables X_1 and X_2 can be expressed by the two other independent random variables y_1 and y_2 by means of a linear transformation, that is,

$$\begin{aligned} X_1 &= g_{11}y_1 + g_{12}y_2 \\ X_2 &= g_{21}y_1 + g_{22}y_2 \end{aligned} \quad (A.27)$$

In matrix form, the above equation is expressed as

$$X = GY$$

where

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

The Fourier transform of a function f is defined as

$$\phi = \int_{-\infty}^{\infty} f e^{iu^T X} dx \quad (A.28)$$

Hence, the Fourier transform of the probability density with respect to y_1 and y_2 is

$$\begin{aligned} \phi_X(u_1, u_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x_1, x_2) e^{iu^T X} dx_1 dx_2 \\ &= E\{\exp(iu^T X)\} \\ &= E\{\exp(iu^T GY)\} \\ &= E\{\exp[i(G^T u)^T Y]\} \\ &= \phi_Y(G^T u) \end{aligned} \quad (A.29)$$

Equation (A.25) gives an expression for the Fourier transform of two-dimensional Gaussian probability density. Then, by substituting $G^T u$ for the variables V in equation (A.28), the Fourier transform of $p(y_1, y_2)$ becomes

$$\begin{aligned}
 \phi_X(u_1, u_2) &= \text{Exp}[i\bar{X}^T(G^T u) - \frac{1}{2}(G^T u)^T \Gamma (G^T u)] \\
 &= \text{Exp}[i(\bar{X}^T G^T) u - \frac{1}{2}u^T (G \Gamma G^T) u] \\
 &= \text{Exp}[i(\bar{Y})^T u - \frac{1}{2}u^T (G \Gamma G^T) u]
 \end{aligned}
 \tag{A.30}$$

It follows that

$$Y = GX$$

and

$$\bar{Y} = G\bar{X} \tag{A.31}$$

and by defining

$$G \Gamma G^T = \Lambda = \text{variance of } X$$

$$\begin{aligned}
 &\begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_2^2 \end{bmatrix} \\
 &= \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_2^2 \end{bmatrix}
 \end{aligned}
 \tag{A.32}$$

the Fourier transform of equation (A.26) becomes

$$\phi_X(u_1, u_2) = \text{Exp}[i\bar{Y}u - \frac{1}{2}u^T \Lambda u] \tag{A.33}$$

Suppose

$$\lambda_1 = -b$$

$$\lambda_2 = -a$$

$$\bar{z}_1 = z_{10} e^{-bt}$$

$$\bar{z}_2 = z_{20} e^{-at}$$

(A.34)

$$\sigma_1^2 = \frac{D}{b} [1 - \exp(-2bt)]$$

$$\sigma_2^2 = \frac{D}{a} [1 - \exp(-2at)]$$

$$\sigma_{12}^2 = \frac{2D}{a+b} \{1 - \exp[-(a+b)t]\}$$

(A.34)

$$\rho = \frac{\sigma_{12}^2}{\sigma_1 \sigma_2}$$

Then equation (A.18) can be written as

$$f = \text{Exp} [i \bar{z}_0^T \zeta - \frac{1}{2} \zeta^T \Lambda \zeta]$$

(A.35)

where

$$\bar{z}_0 = \begin{bmatrix} \bar{z}_{10} \\ \bar{z}_{20} \end{bmatrix}$$

$$\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_2^2 \end{bmatrix}$$

Comparing equation (A.24) with equation (A.27), it is found they are both of the same pattern. Therefore, the function $f(\zeta_1, \zeta_2)$ is actually the Fourier transform of the two-dimensional joint Gaussian probability density, that is

$$P(Z_1, Z_2, t) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{(Z_1 - \bar{Z}_1)^2}{\sigma_1^2} + \frac{(Z_2 - \bar{Z}_2)^2}{\sigma_2^2} - \frac{2\rho}{\sigma_1\sigma_2} (Z_1 - \bar{Z}_1)(Z_2 - \bar{Z}_2) \right] \right\} \quad (A.36)$$

Since the variables Z_1 and Z_2 are defined by the two other variables X and \dot{X} , in the form

$$Z_1 = \dot{X} + aX$$

$$Z_2 = \dot{X} + bX$$

or

$$X = (Z_1 - Z_2) / (a - b)$$

$$\dot{X} = (-bZ_1 + aZ_2) / (a - b)$$

then equation (A.37) can be represented by matrix form as

$$\dot{X} = HZ$$

where

$$X = \begin{bmatrix} X \\ \dot{X} \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & -1 \\ -b & a \end{bmatrix} / (a - b) \quad (A.38)$$

Since $X \rightarrow Z$ transformation is linear, the probability density in X, \dot{X} domain will have the same pattern as the probability

density in Z_1, Z_2 domain with linear transformation for those variances, covariances and expected values of X and \dot{X} .

Therefore, the probability density of X and \dot{X} can be written as

$$p(X, \dot{X}, t) = \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}\sqrt{1-\rho_{XX}^2}} \exp \left\{ -\frac{1}{2(1-\rho_{XX}^2)} \left[\left(\frac{X-X_{10}}{\sigma_X} \right)^2 + \left(\frac{\dot{X}-\dot{X}_{20}}{\sigma_{\dot{X}}} \right)^2 - \frac{2\rho_{XX}}{\sigma_X\sigma_{\dot{X}}} (X-X_{10})(\dot{X}-\dot{X}_{20}) \right] \right\} \quad (A.39)$$

where

$$\begin{bmatrix} \sigma_X^2 & \sigma_{X\dot{X}}^2 \\ \sigma_{X\dot{X}}^2 & \sigma_{\dot{X}}^2 \end{bmatrix} = HGH^T \quad (A.40)$$

and

$$P = \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_2^2 \end{bmatrix} \quad (A.41)$$

$$\rho_{XX} = \frac{\sigma_{X\dot{X}}^2}{\sigma_X\sigma_{\dot{X}}}$$

Expanding (A.40) and equating term by term, we have

$$\sigma_X^2 = \frac{1}{(a-b)^2} [\sigma_1^2 - 2\sigma_{12}^2 + \sigma_2^2] \quad (A.42)$$

$$\begin{aligned}\sigma_X^2 &= \frac{1}{(a-b)^2} [b^2\sigma_1^2 - 2ab\sigma_{12}^2 + a^2\sigma_2^2] \\ \sigma_{XX}^2 &= \frac{1}{(a-b)^2} [(a+b)\sigma_{12}^2 - b\sigma_1^2 - a\sigma_2^2]\end{aligned}\quad (A.42)$$

where

$$a = \frac{1}{2}\beta + i\omega_d$$

$$b = \frac{1}{2}\beta - i\omega_d$$

$$\omega_d = \frac{\sqrt{4\omega_n^2 - \beta^2}}{2}$$

More simply, equation (A.42) can be expressed as

$$\begin{aligned}\sigma_X^2 &= \frac{D}{\omega_n^2\beta} \left[1 - e^{-\beta t} \left(1 + \frac{\beta^2}{2\omega_d^2} \sin\omega_d t - \frac{\beta}{2\omega_d} \sin 2\omega_d t \right) \right] \\ \sigma_X^2 &= D \left[\frac{1}{\beta} - \frac{e^{-\beta t}}{\omega_d t} \left(\frac{\omega_d^2}{\beta} + \frac{\beta}{2} \sin^2\omega_d t - \frac{\omega_d}{2} \sin^2\omega_d t \right) \right] \\ \sigma_{XX}^2 &= \frac{De^{-\beta t}}{\omega_d^2} \sin^2\omega_d t\end{aligned}\quad (A.43)$$

From equation (A.37), the mean values of X and \dot{X} are expressed as

$$\begin{aligned}\langle X_1 \rangle &= \langle (Z_1 - Z_2) / (a-b) \rangle \\ &= (Z_{10}e^{-bt} - Z_{20}e^{-at}) / (a-b) \\ &= \frac{X_{20}}{\omega_d} e^{-\frac{1}{2}\beta t} \sin\omega_d t + \frac{X_{10}}{\omega_d} e^{-\frac{1}{2}\beta t} (\omega_d \cos\omega_d t + \frac{\beta}{2} \sin\omega_d t)\end{aligned}\quad (A.44a)$$

and

$$\begin{aligned}
 \langle \dot{x} \rangle &= \frac{(az_2 - bz_1)}{(a-b)} \\
 &= (az_{20}e^{-at} - bz_{10}e^{-bt}) / (a-b) \quad (A.44b) \\
 &= \frac{x_{20}}{\omega_d} e^{-\frac{1}{2}\beta t} (\omega_d \cos \omega_d t - \frac{\beta}{2} \sin \omega_d t) - \\
 &\quad \frac{\omega_n^2}{\omega_d} x_{10} e^{-\frac{1}{2}\beta t} \sin \omega_d t
 \end{aligned}$$

APPENDIX B

The probability of crossing level y_A between times t and $t+\Delta t$ is defined, as

$$p(X=y_A; t, t+\Delta t) = \int_0^{\infty} \dot{X} p(y_A, \dot{X}) d\dot{X} dt \quad (B.1)$$

where $p(y_A, \dot{X})$ is the joint Gaussian distribution at $X=y_A$. Then, equation (B.1) can be written as

$$p(X=y_A; t, t+\Delta t) = \frac{\exp \left[-\frac{(y_A - \bar{X})^2}{2\sigma_X^2} \right]}{2\pi\sigma_X\sigma_{\dot{X}}\sqrt{1-\rho_{XX}^2}} \int_0^{\infty} \dot{X} \times \exp \left[-\frac{\left[\dot{X} - \left(\bar{\dot{X}} + \frac{\sigma_{XX}^2}{\sigma_X^2} (y_A - \bar{X}) \right) \right]^2}{2(1-\rho_{XX}^2)\sigma_{\dot{X}}^2} \right] d\dot{X} dt \quad (B.2)$$

Defining

$$K = \frac{\exp \left[-\frac{(y_A - \bar{X})^2}{2\sigma_X^2} \right]}{2\pi\sigma_X\sigma_{\dot{X}}\sqrt{1-\rho_{XX}^2}}$$

and

$$A = \dot{X} + \frac{\sigma_{XX}^2}{\sigma_X^2} (y_A - \bar{X})$$

$$\sigma_A^2 = (1-\rho_{XX}^2)\sigma_{\dot{X}}^2$$

equation (B.2) is rewritten as

$$p(X=y_0; t, t+\Delta t) = K \int_0^{\infty} \dot{X} \exp\left[-\frac{(\dot{X}-A)^2}{2\sigma_A^2}\right] d\dot{X} dt$$

$$= dt \cdot K \left\{ \int_0^{\infty} (\dot{X}-A) \exp\left[-\frac{(\dot{X}-A)^2}{2\sigma_A^2}\right] d(\dot{X}-A) + \right.$$

$$\left. \int_0^{\infty} A \exp\left[-\frac{(\dot{X}-A)^2}{2\sigma_A^2}\right] d(\dot{X}-A) \right\}$$

$$= dt \cdot K \left[\int_{-A}^{\infty} u \exp(-u^2/2\sigma_A^2) du + \right.$$

$$\left. \int_{-A}^{\infty} A \exp(-u^2/2\sigma_A^2) du \right]$$

(B.3)

$$= dt \cdot K \left[\int_0^{\infty} u \exp(-u^2/2\sigma_A^2) du - \right.$$

$$\left. \int_0^A u \exp(-u^2/2\sigma_A^2) du + A \int_0^{\infty} \exp(-u^2/2\sigma_A^2) du + \right.$$

$$\left. A \int_0^A \exp(-u^2/2\sigma_A^2) du \right]$$

$$= K \cdot dt \left[\sigma_A^2 + A\sqrt{\pi/2}\sigma_A - 2\sigma_A^2 \int_0^{A/\sqrt{2}\sigma_A} u \exp(-u^2) du \right.$$

$$\left. + \sqrt{2}A\sigma_A^2 \int_0^{A/\sqrt{2}\sigma_A} \exp(-u^2) du \right]$$

$$= K \cdot dt \left[\sigma_A^2 + A\sqrt{\pi/2}\sigma_A - 2\sigma_A^2 g\left(\frac{A}{\sqrt{2}\sigma_A}\right) + \right.$$

$$\left. \sqrt{2}A\sigma_A^2 h\left(\frac{A}{\sqrt{2}\sigma_A}\right) \right]$$

APPENDIX C

Section C-1

To solve the polynomial

$$S^4 + a_1 S^3 + a_2 S^2 + a_3 S + a_4 = 0 \quad (C.1)$$

let

$$\begin{aligned} S^4 + a_1 S^3 + a_2 S^2 + a_3 S + a_4 &= (S^2 + A_1 S + A_2)(S^2 + B_1 S + B_2) \\ &= S^4 + (A_1 + B_1)S^3 + (A_1 B_1 + A_2 + B_2)S^2 + (A_2 B_1 + A_1 B_2)S \\ &\quad + A_2 B_2 \end{aligned}$$

Then,

$$A_1 + B_1 = a_1 \quad (C.2)$$

$$A_1 B_1 + A_2 + B_2 = a_2 \quad (C.3)$$

$$A_2 B_1 + A_1 B_2 = a_3 \quad (C.4)$$

$$A_2 B_2 = a_4 \quad (C.5)$$

Substituting $a_1 - B_1 = A_1$ into equation (C.4),

$$A_2 B_1 + B_2 a_1 - B_2 B_1 = a_3 \quad (C.6)$$

or

$$B_1(A_2 - B_2) = a_3 - B_2 a_1$$

Equations (C.3) to (C.6) are now rearranged in the form

$$\begin{aligned} A_2 &= a_4/B_2 \\ B_1 &= (a_3 - B_2 a_1)/(A_2 - B_2) \\ A_1 &= (a_2 - A_2 - B_2)/B_1 \\ a_1 - A_1 - B_1 &= 0 \end{aligned} \quad (C.7)$$

Assume a value of B_2 to obtain A_2 , then B_1 and A_1 , and then check whether $a_1 - A_1 - B_1 = 0$. If not, increase or decrease B_2 .

Section C-2

To obtain the partial fraction of a function

$$\frac{u_1 S^3 + u_2 S^2 + u_3 S + u_4}{(s^2 + as + b)(s^2 + cs + d)} \quad (C.8)$$

Let

$$\frac{u_1 S^3 + u_2 S^2 + u_3 S + u_4}{(S^2 + as + b)(S^2 + cs + d)} = \frac{V_1 S + V_2}{S^2 + as + b} + \frac{V_3 S + V_4}{S^2 + cs + d} \quad (C.9)$$

Then,

$$\begin{aligned} u_1 S^3 + u_2 S^2 + u_3 S + u_4 &= (V_1 + V_2) S^3 + (V_1 c + V_2 + V_3 a + V_4) S^2 \\ &\quad + (V_1 d + V_2 c + V_3 b + V_4 a) S + (V_2 d + V_4 b) \end{aligned} \quad (C.10)$$

Then,

$$v_1 + v_3 = u_1 \quad (C.11)$$

$$v_1 e + v_2 + v_3 a + v_4 = u_2 \quad (C.12)$$

$$v_1 d + v_2 c + v_3 b + v_4 a = u_3 \quad (C.13)$$

$$v_2 d + v_4 b = u_4 \quad (C.14)$$

From equations (C.11) and (C.14),

$$v_1 = u_1 - v_3 \quad (C.15)$$

$$v_2 = \frac{u_4}{d} - \frac{v_4 b}{d}$$

Substituting equation (C.15) into equations (C.12) and (C.13), we have

$$cu_1 + \frac{u_4}{d} + v_3(a-c) + v_4(1-\frac{b}{d}) = u_2$$

$$\text{or } du_1 + \frac{c}{d}u_4 + v_3(b-d) + v_4(a-\frac{cb}{d}) = u_3$$

$$\text{or } v_3 + v_4(1-b/d)/(a-c) = (u_2 - cu_1 - \frac{u_4}{d})/(a-c)$$

$$\text{or } v_3 + v_4(a-\frac{cb}{d})/(b-d) = (u_3 - du_1 - \frac{c}{d}u_4)/(b-d)$$

$$\text{or } v_4 = \left[\frac{(1-b/d)}{(a-c)} \cdot \frac{(a-cb/d)}{(b-d)} \right]^{-1} \left[\frac{u_2 - cu_1 - u_4}{(a-c)} - \frac{u_3 - du_1 - \frac{c}{d}u_4}{(b-d)} \right]$$

So,

$$V_4 = \frac{\left[\frac{u_2 - cu_1 - u_4}{(a-c)} - \frac{u_3 - du_1 - \frac{c}{d}u_4}{(b-d)} \right]}{\left[\frac{(1-b/d)}{(a-c)} - \frac{(a-cb/d)}{(b-d)} \right]}$$

$$V_3 = \frac{(u_2 - cu_1 - u_4/d) - V_4(1 - \frac{b}{d})}{(a-c)} \quad (C.16)$$

$$V_2 = \frac{u_4}{d} - \frac{V_4 b}{d}$$

$$V_1 = u_1 - V_3$$

Section C-3

To obtain the solution of the Laplace transform of the function

$$\frac{V_1 s + V_4}{(s + v_1 \mu_1)^2 + \omega_1^2} + \frac{V_3 s + V_4}{(s + v_2 \mu_2)^2 + \omega_2^2} \quad (C.17)$$

Let the inverse Laplace transform of equation (C.17) be written as

$$f(t) = L^{-1} \left[\frac{V_1 s + V_2}{(s + v_1 \mu_1)^2 + \omega_1^2} + \frac{V_3 s + V_4}{(s + v_2 \mu_2)^2 + \omega_2^2} \right] \quad (C.18)$$

By defining

$$r_1 = V_1$$

$$r_2 = (V_2 - r_1 \beta_1) / \omega_1 \quad (C.19)$$

$$r_3 = v_3$$

(C.19)

$$r_4 = (v_4 - r_3 \zeta_2 \beta_2) / \omega_2$$

Equation (C.18) can be expressed as

$$f(t) = L^{-1} \left[r_1 \frac{(s + \zeta_1 \beta_1)}{(s + \zeta_1 \beta_1)^2 + \omega_1^2} + r_2 \frac{\omega_1^2}{(s + \zeta_1 \beta_1)^2 + \omega_1^2} + r_3 \frac{(s + \zeta_2 \beta_2)}{(s + \zeta_2 \beta_2)^2 + \omega_2^2} + r_4 \frac{\omega_2^2}{(s + \zeta_2 \beta_2)^2 + \omega_2^2} \right] \quad (C.20)$$

$$= \exp[-\zeta_1 \beta_1 t] (r_1 \cos \omega_1 t + r_2 \sin \omega_1 t) + \exp[-\zeta_2 \beta_2 t] (r_3 \cos \omega_2 t + r_4 \sin \omega_2 t)$$

More simply, let the phase angles ϕ_1 and ϕ_2 be defined as

$$\phi_1 = \tan^{-1} \left(\frac{r_1}{r_2} \right)$$

(C.20)

$$\phi_2 = \tan^{-1} \left(\frac{r_3}{r_4} \right)$$

Equation (C.20) can be expressed as

$$f(t) = \exp[-\zeta_1 \beta_1 t] \sin(\omega_1 t + \phi_1) + \exp[-\zeta_2 \beta_2 t] \sin(\omega_2 t + \phi_2) \quad (C.21)$$

FORTRAN COMPUTER PROGRAMS FOR
NUMERICAL RESULTS

C

PROGRAM SETLE(INPUT,OUTPUT)
 THE CALCULATIONS OF THE SETTLING TIME
 DIMENSION PA(650)
 PRINT 154

154 FOR A1(1:10)
 5 READ A,X01,X02,W0,ZE
 B=2.*ZE*W0
 D=1.
 IF (W0-..).2,4,3
 3 S=3.14159265

1 FOR A1(1:10:0)
 N=1000
 ZE=N/(2.*W0)

PRINT 1,X01,X02,W0,ZE

11 FORMAT(20X,*X01=*,F5.3,10X,*X02=*,F5.3,10X,*W0=*,F5.2,10X,*ZE=*,
 IF5.3,/,//)

WS=W0*W0

N=SQRT(ABS(4*W0*W0-B*B))/2

PHI=ATAN(X02/(X02*B/(2.*N)+WS*X01/W))

Y=0.05*X01

DO 100 I=1,N

T=(1-I)*S+PHI)/W

CC=EXP(-B*T/2)

XB1=(X02*CC*SIN(W*T)+X01*CC*(W*COS(W*T)+B*SIN(W*T)/2.))/W

IF(ABS(XB1).LE.Y) GO TO 110

100 CONTINUE

110 T=1-0.01

CC=EXP(-B*T/2)

XB1=(X02*CC*SIN(W*T)+X01*CC*(W*COS(W*T)+B*SIN(W*T)/2.))/W

IF(ABS(XB1)-Y)110,113,113

113 PRINT 15,I,XB1

15 FORMAT(20X,*THE SETTLING TIME IS =*,F10.5,*AND XB1 IS =*,F10.5,/,//)

TF=T

S=2.*3.141596

YA=.20

WW=W*W

NS=W0*W0

DT=0.01

PRO=0

PA(1)=0

DO 111 I=2,N

T=0.01*I

CC=EXP(-B*T/2)

E=EXP(-B*T)

S1=B*B*SIN(W*T)*SIN(W*T)/2.

S2=B*W*SIN(2.*W*T)/2.

X1=D*(1.-E*(WW+S1+S2)/WW)/(WS*B)

X2=D*(1.-E*(WW+S1+S2)/WW)/B

XX=SQRT(D*E)*SIN(W*T)/W

XX=XX*XX

XB1=(X02*CC*SIN(W*T)+X01*CC*(W*COS(W*T)+B*SIN(W*T)/2.))/W

XB2=(X02*CC*(W*COS(W*T)-B*SIN(W*T)/2.)-WS*X01*CC*SIN(W*T))/W

RO=XX*XX/(X1*X2)

RS=SQRT(ABS(1-RO))

YY=(YA-XB1)*(YA-XB1)



```

RK=EXP(-YY/(2.*X1))/(S*RS*SQR(X1*X2))
A=XB2+XA*(YA-XB1)/X1
XA=RS*RS*AZ
Z1=A/SQR(2.*AA)
Z1=ABS(Z1)
IF(Z1-2.900)120,121,121
121 G7=0.5
H2=0.886226925
G0 TO 123
120 G2=0.0072338284 -0.0797777009*Z1 +0.82571971*Z1**2
1-0.53941503*Z1**3,+0.088555368*Z1**4+0.018628668*Z1**5-
20.004610728*Z1**6
H7=0.0007730240 +0.98005269*Z1 +0.12772535*Z1**2.
1-0.6590164*Z1**3+0.37857007*Z1**4-0.09034195*Z1**5+
20.0080652865*Z1**6
123 PY=RK*(XA+A*SQR(5*AA/4.))-2*XA*G2+SQR(2.)*A*XA*H2
PA(1)=ABS(PY)
PRQ=PRQ+(PA(1)+PA(1-1))/2.*DT
IF(T-TT)111,200,200
111 CONTINUE
200 PF=1-EXP(-PRQ)
PRINT 211,PF,T
211 FORMAT(10X,'THE FIRST PASSAGE PROB. =*,F10.5,*AT TIME =*,F10.5,///)
G0 TO 5
2 STOP
END

```



```
PROGRAM FIRST(INPUT,OUTPUT)
DIMENSION PX(650)
X01=0.2
X02=0.1
```

```
S=2.*3.141596
21 PFAD.1,W0,D,B,YA
1 FORMAT(4F10.0)
IF(W0-0)20,20,22
22 PRINT 11.0,W0,B,YA
11 FORMAT(20X,*D=*,F5.2,10X,*Q=*,F5.2,10X,*B=*,F5.2,10X,*YA=*,F10.5)
```

```
N=600
W=SQRT(4.*W0*W0-R*B)/2.
WW=W*W
WS=W0*W0
I=0
DT=0.01
```

```
PRO=0
PX(1)=0
DO 100 I=2,N
T=0.01*I
E=EXP(-R*T)
S1=R*B*SIN(W*T)*SIN(W*T)/2.
```

```
S2=R*W*SIN(2.*W*T)/2.
X1=0*(1.-E*(WW+S1+S2)/WW)/(WS*R)
X2=0*(1.-E*(WW+S1-S2)/WW)/R
XX=SQRT(D*E)*SIN(W*T)/W
XX=XX*XX
XR1=(X02*CC*SIN(W*T)+X01*CC*(W*COS(W*T)+B*SIN(W*T)/2.))/W
```

```
XR2=(X02*CC*(W*COS(W*T)-R*SIN(W*T)/2.))-W*X01*CC*SIN(W*T))/W
RO=XX*XX/(X1*X2)
RS=SQRT(1.-PO)
YY=(YA-XR1)*(YA-XR1)
RK=EXP(-YY/(2.*X1))/(2.*S*Y1*X2*RS)
A=XR2+XX*(YA-XR1)/X1
```

```
XA=RS*PS*X2
Z1=A/SQRT(2.*XA)
IF(71-2.900)120,121,121
121 G7=0.5
H7=0.886226925
GO TO 123
```

```
120 GZ=0.0042338284*Z1**6-0.0707777009*Z1**5.+0.82571971*Z1**4
1-0.53941503*Z1**3.+0.08855368*Z1*Z1+0.016628668*Z1-0.004610728
H7=0.0007735246*Z1**6+0.98005269*Z1**5+0.12772535*Z1**4
1-0.6590164*Z1**3+0.37857007*Z1*Z1-0.09034195*Z1+0.0080632865
123 PY=PK*(XA+A*SQRT(S*YA/2.))-X*XA*GZ+SQRT(2.)*A*XA*HZ)
```

```
T=T+DT
PX(I)=PY
PRO=PRO+(PX(I)+PX(I-1))/2.*DT
PF=1-EXP(-PRO)
PRINT 10,I,PT,PRO,PF
10 FORMAT(10X,F15.6,10X,F15.6,10X,F15.6,10X,F15.6)
```

```
100 CONTINUE
GO TO 21
20 STOP
END
```



```

PROGRAM OPTIM (INPUT,OUTPUT)
DIMENSION A(50),D(50),
DIMENSION F(50),G(50),C(50),RR(50),RI(50)
DIMENSION X(100)

```

```

DO 511 KJ=1,2
101 PRINT 11
11 FORMAT(1H1)
M2=1.
DO 14 J=1,10
W1=0.3*J.
SM1=1.
SM2=.2*SM1
SK1=W1*W1*SM1
C1=0.1*W1*SM1*KJ
C2=0.1*SM2*W2*KJ
SK2=W2*W2*SM2
D(1)=1
D(2)=(C1+C2)/SM1+C2/SM2
D(3)=(SK1+SK2)/SM1+SK2/SM2+C1*C2/(SM1*SM2)
D(4)=C1*SK2/(SM1*SM2)+C2*SK1/(SM1*SM2)
D(5)=SK1*SK2/(SM1*SM2)
D1=C2/SM2
J2=SK2/SM2
PRINT 10,SM1,C1,SK1,SM2,C2,SK2,D1,D2
10 FORMAT(5X,*M=*,F10.5,5X,*C1=*,F10.5,5X,*SK1=*,F10.5,/,5X,*M2=*,
F10.5,5X,*C2=*,F10.5,5X,*SK2=*,F10.5,/,5X,*D1=*,F10.5,5X,
2*D2=*,F10.5,///)
Z1=C1/(2.*SM1*W1)
W2=SQR(5F2/SM2)
Z2=C2/(2.*SM2*W2)
WD1=W1*SQR(1.-Z1*Z1)
WD2=W2*SQR(1.-Z2*Z2)
PRINT 15,W1,Z1,WD1,W2,Z2,WD2
15 FORMAT(5X,*W1=*,F10.5,5X,*Z1=*,F10.5,5X,*WD1=*,F10.5,/,
5X,*W2=*,F10.5,5X,*Z2=*,F10.5,5X,*WD2=*,F10.5,///)
N=4
X(1)=1
X(2)=D(2)*D(2)-2.*D(3)
X(3)=D(3)*D(3)+2.*D(5)-2.*D(2)*D(4)
X(4)=D(4)*D(4)-2.*D(3)*D(5)
X(5)=D(5)*D(5)
CALL FAST(X,RR,RI,N)
DO 150 I=1,10
F(1)=0
150 CONTINUE
GH=D2*D2
F(2)=2.*D2*D1*D1
F(4)=1.
C(1)=ABS(RR(1))
G(1)=ABS(W1(1))
C(2)=C(1)
G(2)=-G(1)
C(3)=-C(1)
G(3)=G(1)
C(4)=-C(1)

```



```
G(4)=-G(1)
C(5)=ABS(RR(3))
G(5)=ABS(RI(3))
C(6)=C(5)
G(6)=-G(5)
C(7)=-C(5)
G(7)=G(5)
C(8)=-C(5)
G(8)=-G(5)
CALL RESIDUE(C,G,F,GH,GB3,GB4)
```

```
GA1=GB4
F(2)=GH
F(4)=2.*D2+D1*D1
F(6)=1.
GH=0
CALL RESIDUE(C,G,F,GH,GB3,GB4)
```

```
GA2=GB4
F(2)=D1*D1
GH=D2*D2
F(4)=0
F(6)=0
CALL RESIDUE(C,G,F,GH,GB3,GB4)
```

```
GA3=GB4
F(2)=GH
F(4)=D1*D1
F(6)=0
GH=0
CALL RESIDUE(C,G,F,GH,GB3,GB4)
```

```
GA4=GB4
GN1=SQRT(GA2/GA1)/2./3.14159265
GN2=SQRT(GA4/GA3)/2./3.14159265
PRINT 512,GN1,GN2
```

```
512 FOR IAT(5X,F15.8,5X,F15.8)
```

```
14 CONTINUE
```

```
511 CONTINUE
```

```
STOP
```

```
END
```



```

SUBROUTINE FAST(AR,RR,RI,N)
DIMENSION AR(50),A1(50),BR(50),B1(50),RR(50),R1(50),CR(50),CI(50)
DIMENSION PHI(50)
AI(1)=0
AI(2)=0
AI(3)=0
AI(4)=0
AI(5)=0
PI=3.1415926535
NN=N+1
DIFF=0.00000001
NM=N
BR(1)=AR(1)
B1(1)=A1(1)
CR(1)=AR(1)
CI(1)=A1(1)
DO 10 K=1,NM
X=0
Y=1.
2 DO 20 I=2,NN
BR(I)=AR(I)+BR(I-1)*X-B1(I-1)*Y
B1(I)=A1(I)+BR(I-1)*Y+B1(I-1)*X
20 CONTINUE
DO 30 I=2,N
CR(I)=BR(I)+CR(I-1)*X-CI(I-1)*Y
CI(I)=B1(I)+CR(I-1)*Y+CI(I-1)*X
30 CONTINUE
D=CR(N)*CR(N)+CI(N)*CI(N)
IF (D=0) 113,112,113
112 D=0.001
113 X=A-(BR(NN)*CR(N)+B1(NN)*CI(N))/D
Y=Y+(BR(NN)*CI(N)-B1(NN)*CR(N))/D
IF (BR(NN)*BR(NN)+B1(NN)*B1(NN)-DIFF) 3,3,2
3 NN=NN+1
N=NN-1
DO 40 I=2,NN
AR(I)=BR(I)
A1(I)=B1(I)
40 CONTINUE
IF (ABS(Y/X)-0.00010) 4,5,5
4 Y=Y/2
5 RR(K)=X
R1(K)=Y
IF (RR(K)=0) 26,15,26
26 PHI(K)=ATAN(ABS(R1(K)/RR(K)))
IF (RR(K)=0) 16,15,21
15 IF (R1(K)=0) 17,60,19
17 PHI(K)=0.5*PI
GO TO 60
19 PHI(K)=0.5*PI
GO TO 60
16 IF (R1(K)=0) 22,61,23
22 PHI(K)=PI+PHI(K)
GO TO 60
23 PHI(K)=PI-PHI(K)

```



ROUTINE FAST

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GO TO 60
21 IF (RI(K)-0) 24, 62, 25
24 PHI(K)=-PHI(K)
GO TO 60

61 PHI(K)=-PI

GO TO 60

62 PHI(K)=0

GO TO 60

25 PHI(K)=PHI(K)

60 AR1=(RR(K)*RR(K)+RI(K)*RI(K))*0.25*COS(PHI(K)/2)

AI1=(RR(K)*RR(K)+RI(K)*RI(K))*0.25*SIN(PHI(K)/2)

AR2=-(RR(K)*RR(K)+RI(K)*RI(K))*0.25*COS(PHI(K)/2)

AI2=-(RR(K)*RR(K)+RI(K)*RI(K))*0.25*SIN(PHI(K)/2)

PRINT 500, RR(K), RI(K)

PRINT 60, AR1, AI1, AR2, AI2

65 FORMAT(5X, F15.6, 5X, F15.6, /, 5X, F15.6, 5X, F15.6)

RR(K)=0

RI(K)=0

10 CONTINUE

100 FORMAT(I3)

110 FORMAT(6F10.0)

120 FORMAT(8F10.0)

500 FORMAT(50X, F15.6, F15.6)

RETURN

END



SUBROUTINE RESIDUE(C,D,F,GH,GB3,GB4)

DIMENSION A(40),B(40),C(40),D(40),AF(40),BF(40),R(40),PHI(40)

DIMENSION F(50)

N=4

GB4=0

J=1

PI=3.1415926535

115 IF(C(1)-0) 20,15,20

26 PHI(J)=ATAN(ABS(D(1)/C(1)))

IF(C(1)-0) 16,15,21

15 IF(D(1)-0) 17,20,19

17 PHI(J)=-0.5*PI

GO TO 20

19 PHI(J)=0.5*PI

GO TO 20

16 IF(D(1)-0) 22,20, 23

22 PHI(J)=PI+PHI(J)

GO TO 20

23 PHI(J)=PI-PHI(J)

GO TO 20

21 IF(D(1)-0) 24,20, 25

24 PHI(J)=-PHI(J)

GO TO 20

25 PHI(J)=PHI(J)

20 R(J)=SQRT(C(1)*C(1)+D(1)*D(1))

NN=NN+1

X=0

Y=0

DO 100 I=1,NN

A(I)=C(I)-C(I+1)

B(I)=D(I)-D(I+1)

100 CONTINUE

MN=NN-1

DO 110 I=1,MN

AA=A(I)*A(I+1)+B(I)*B(I+1)

AB=A(I)*B(I+1)+A(I+1)*B(I)

A(I+1)=AA

B(I+1)=AB

110 CONTINUE

AF(I)=R(J)*COS(PHI(J))

AF(2)=R(J)**2*COS(PHI(J)*2)

AF(3)=R(J)**3*COS(PHI(J)*3)

AF(4)=R(J)**4*COS(PHI(J)*4)

AF(5)=R(J)**5*COS(PHI(J)*5)

AF(6)=R(J)**6*COS(PHI(J)*6)

AF(7)=R(J)**7*COS(PHI(J)*7)

AF(8)=R(J)**8*COS(PHI(J)*8)

AF(9)=R(J)**9*COS(PHI(J)*9)

AF(10)=R(J)**10*COS(PHI(J)*10)

BF(1)=R(J)*SIN(PHI(J))

BF(2)=R(J)**2*SIN(PHI(J)*2)

BF(3)=R(J)**3*SIN(PHI(J)*3)

BF(4)=R(J)**4*SIN(PHI(J)*4)

BF(5)=R(J)**5*SIN(PHI(J)*5)

BF(6)=R(J)**6*SIN(PHI(J)*6)



BF(7)=R(J)**7*SIN(PHI(J)*7)
BF(9)=R(J)**9*SIN(PHI(J)*9)
BF(8)=R(J)**8*SIN(PHI(J)*8)
BF(10)=R(J)**10*SIN(PHI(J)*10)

GO TO 1-10

X=X+AF(1)*F(1)

Y=Y+BF(1)*F(1)

51 CONTINUE

X=X+GH

BB=SQRT(AA*AA+AB*AB)

GF=SQRT(A*X+Y*Y)

GB=GF/BB

IF(X-0.786,75,86)

86 PHIR=ATAN(ABS(Y/X))

IF(X<0)/6,75,81

75 IF(Y<0)/7,86,79

77 PHIR=-.5*PI

GO TO 80

79 PHIR=.5*PI

GO TO 80

76 IF(Y<0)/2,207,83

82 PHIR=PI+PHIR

GO TO 80

83 PHIR=PI-PHIR

GO TO 80

81 IF(Y<0)/4,206,85

84 PHIR=-PHIR

GO TO 80

206 PHIR=0

GO TO 80

207 PHIR=-PI

GO TO 80

85 PHIR=PHIR

80 PHI=180.*PHIR/PI

IF(AA<0)/46,35,40

40 PHRI=ATAN(ABS(AB/AA))

IF(AA<0)/36,35,41

35 IF(AB<0)/37,40,39

37 PHRI=-.5*PI

GO TO 40

39 PHRI=.5*PI

GO TO 40

36 IF(AB<0)/42,209,43

42 PHRI=PI+PHRI

GO TO 40

43 PHRI=PI-PHRI

GO TO 40

41 IF(AB<0)/44,208,45

44 PHRI=-PHRI

GO TO 40

208 PHRI=0

GO TO 40

209 PHRI=PI

GO TO 40

45 PHRI=PHRI



40 PH2=180.*PHR1/PI
PH3=PH1K-PHR1
GB1=GB* \cos (PH3)
GB2=GB* \sin (PH3)

PRINT 52,GB1,GB2,PH3

52 FORMAT(15X,F15.8,5X,F15.8,5X,F15.8)

GB3=-GB2*2

GB4=GB4+GB3

PRINT 215,GB3,GB4

215 FORMAT(10X,*RESIDUE =*,F15.6,5X,*TOTAL RESIDUE=*,F15.5)

IF((J+1).EQ.N)GO TO 60

J=J+2

CC=C(1)

DD=D(1)

C(1)=C(J)

D(1)=D(J)

C(J)=CC

D(J)=DD

GO TO 115

60 RETURN

END

